UC-Irvine Generalized Ricci Flow Learning Seminar

Flows of G_2 Structures Associated to Calabi–Yau Manifolds

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Overview

Goal

Establish a correspondence between the Laplacian flow and coflow on torus bundles over Calabi–Yau 2- and 3-folds with Monge–Ampère flows on the base.

This is joint work with Sébastien Picard.

Main Results

Theorem (Picard–S.)

Start the Laplacian flow with initial data

$$\varphi = -dr \wedge dr^2 \wedge dr^3 + dr^1 \wedge \omega + dr^2 \wedge Re\,(\Upsilon) + dr^3 \wedge Im\,(\Upsilon) \text{ on } T^3 \times X^4,$$

or

$$\varphi = Re\,(\Upsilon) - dr \wedge \omega \text{ or } S^1 \times X^6.$$

Then the Laplacian flow exists for all time t *and is given by the* $MA^{\frac{1}{3}}$ *flow (up to diffeomorphism) and converges to a stationary point*

$$\begin{aligned}\varphi_\infty &= -dr \wedge dr^2 \wedge dr^3 + dr^1 \wedge \Theta_\infty^* \omega_{CY} \\ &\quad + dr^2 \wedge Re\,(\Theta_\infty^* \Upsilon) + dr^3 \wedge Im\,(\Theta_\infty^* \Upsilon) \text{ on } T^3 \times X^4\end{aligned}$$

or

$$\varphi_\infty = Re\,(\Theta_\infty^* \Upsilon) - dr \wedge \Theta_\infty^* \omega_{CY} \text{ on } S^1 \times X^6,$$

where Θ_∞ *is a diffeomorphism on the base and* ω_{CY} *is the unique Ricci-flat Kähler metric in the class* $[\omega]$.

Main Results

Theorem (Picard–S.)

Start the Laplacian coflow with initial data

$$\psi = -2^{-\frac{4}{3}} \cdot \frac{1}{2}\omega^2 + 2^{-\frac{4}{3}} \cdot dr^2 \wedge dr^3 \wedge \omega$$
$$+ 2^{\frac{2}{3}} \cdot dr^3 \wedge dr^1 \wedge Re\,(\Upsilon) + 2^{\frac{2}{3}} \cdot dr^1 \wedge dr^2 \wedge Im\,(\Upsilon) \text{ on } T^3 \times X^4$$

or

$$\psi = -2 \cdot dr \wedge Im\,(\Upsilon) - \frac{1}{4} \cdot \frac{1}{2}\omega^2 \text{ on } S^1 \times X^6.$$

Then the Laplacian coflow exists for all time t and is given by the Kähler–Ricci flow (up to diffeomorphism) and converges to a stationary point

$$\psi_\infty = -2^{-\frac{4}{3}} \cdot \frac{1}{2}\Theta_\infty^*\omega_{CY}^2 + 2^{-\frac{4}{3}} \cdot dr^2 \wedge dr^3 \wedge \Theta_\infty^*\omega_{CY}$$
$$+ 2^{\frac{2}{3}} \cdot dr^3 \wedge dr^1 \wedge Re\,(\Theta_\infty^*\Upsilon) + 2^{\frac{2}{3}} \cdot dr^1 \wedge dr^2 \wedge Im\,(\Theta_\infty^*\Upsilon) \text{ on } T^3 \times X^4$$

or

$$\psi_\infty = -2 \cdot dr \wedge Im\,(\Theta_\infty^*\Upsilon) - \frac{1}{4} \cdot \frac{1}{2}\Theta_\infty^*\omega_{CY}^2 \text{ on } S^1 \times X^6,$$

where Θ_∞ is a diffeomorphism on the base and ω_{CY} is the unique Ricci-flat Kähler metric in the class $[\omega]$.

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Calabi–Yau Manifolds

Definition

A (Kähler) Calabi–Yau n-fold X is a complex manifold with dimension n (and real dimension $2n$) admitting:

- a Kähler metric ω,
- and a nowhere vanishing holomorphic $(n, 0)$-form Υ.

Throughout, we will refer to the pair (ω, Υ) as a (Kähler) Calabi–Yau structure.

A Calabi–Yau manifold has the following properties:

- the canonical bundle K_X is trivial,
- the first Chern class $c_1(X)$ vanishes,
- the Ricci-form $\mathrm{Ric}(\omega, J)$ is given by $2i\partial\overline{\partial}(\log|\Upsilon|_\omega)$ and it vanishes if and only if $|\Upsilon|_\omega$ is constant.

Yau's Theorem

Theorem (Yau)

Let (X, ω) be a compact Kähler manifold with $c_1(X) = 0$ and let $F\colon X \to \mathbb{R}$ be a function such that

$$\int_X e^F \omega^n = \int_X \omega^n.$$

Then there is a smooth function $u\colon X \to \mathbb{R}$, unique up to the addition of a constant, such that

$$\omega + i\partial\bar{\partial}u > 0 \text{ and } (\omega + i\partial\bar{\partial}u)^n = e^F \omega^n.$$

Yau's theorem implies the existence of a Ricci-flat Kähler metric in the cohomology class $[\omega]$.

This metric is unique in its Kähler class. When X is a Calabi–Yau manifold, we denote it by ω_{CY} and refer to it as a Calabi–Yau metric.

Yau's theorem and its proof involved the solving of complex Monge–Ampère equations, which have since been studied extensively.

Monge–Ampère Flows

On a compact Kähler manifold, there is a class of flows of Kähler metrics that are related to complex Monge–Ampère equations.

Theorem (Picard–Zhang)

Let (X, ω) be a compact Kähler manifold. Let $a\colon X \to \mathbb{R}$ be a function and let $H\colon \mathbb{R}^+ \to \mathbb{R}$ be a smooth function with $H' > 0$. Then there exists a solution u_t to the parabolic complex Monge–Ampère equation

$$\frac{\partial}{\partial t} u_t = H\Big(e^{-a}\frac{\det(\omega + i\partial\bar{\partial}u_t)}{\det \omega}\Big), \qquad \omega + i\partial\bar{\partial}u_t > 0, \qquad u_0 = 0.$$

This solution exists for all time t. Moreover, the metrics $\widetilde{\omega}_t = \omega + i\partial\bar{\partial}u_t$ converge in each $C^k(X, g)$-norm to a limiting metric $\omega' \in [\omega]$.

When X is a Calabi–Yau manifold, the limiting metric is the Calabi–Yau metric ω_{CY}.

Monge–Ampère Flows

Certain choices of the functions a and H give familiar special cases:

- Kähler–Ricci flow ($H(\rho) = \log \rho$),
- Anomaly flow ($H(\rho) = \rho$).

We have two particular cases of importance:

- MA$^{\frac{1}{3}}$ flow ($a = 2\log|\Upsilon|_\omega$, $H = 6K\rho^{\frac{1}{3}}$):

$$\frac{\partial}{\partial t}u_t = 6K\left(e^{-2\log|\Upsilon|_\omega}\frac{\det(\omega + i\partial\bar{\partial}u_t)}{\det\omega}\right)^{\frac{1}{3}},$$

- Kähler–Ricci flow ($a = 2\log|\Upsilon|_\omega$, $H = 2K\log\rho$):

$$\frac{\partial}{\partial t}u_t = 2K\log\left(\frac{\det(\omega + i\partial\bar{\partial}u_t)}{\det\omega}\right) - 2K\log|\Upsilon|^2_\omega.$$

Uniform Estimates

The evolving metrics $\widetilde{g}_t$ from a Monge–Ampère flow satisfy uniform estimates: There exist positive constants C and C_k such that

$$C^{-1} \cdot g \leq \widetilde{g}_t \leq C \cdot g \text{ and } |\nabla_g^k \widetilde{\omega}_t|_g \leq C_k.$$

We also have exponential convergence of the flow: There exist positive constants C_k and λ_k such that

$$\left| \frac{\partial}{\partial t} \nabla_g^k \widetilde{\omega}_t \right|_g \leq C_k e^{-\lambda_k t}.$$

These estimates were previously known for certain special cases like the Kähler–Ricci flow (Cao, Phong–Sturm).

Structures from the Octonions

Let $\mathbb{O}$ denote the normed division algebra of the octonions.

We have the commutator $[\cdot,\cdot]$ and associator $[\cdot,\cdot,\cdot]$ on $\mathbb{O}$:

$$[a,b] = ab - ba,$$

$$[a,b,c] = (ab)c - a(bc).$$

Using these forms, we can define a 3-form φ and a 4-form ψ on $Im\,\mathbb{O}$:

$$\varphi(a,b,c) = \frac{1}{2}\langle a,[b,c]\rangle,$$

$$\psi(a,b,c,d) = \frac{1}{2}\langle a,[b,c,d]\rangle.$$

Additionally, we have an octonionic cross-product $\times$ on $Im\,\mathbb{O}$:

$$a \times b = Im\,(ab).$$

The Group G_2

We can identify $\mathbb{R}^7$ with $Im\,\mathbb{O}$ and endow it with all the aforementioned structures. Together, we get the standard G_2 structure on $\mathbb{R}^7$ which consists of:

- the standard Euclidean metric g_0;
- the standard orientation and standard volume form $\mu_0 = e^1 \wedge \ldots \wedge e^7$ associated to g_0, where $e^1, \ldots, e^7$ is the standard ON basis;
- the associative 3-form φ_0;
- the coassociateive 4-form ψ_0;
- the octonionic cross-product $\times_0$.

One can check that $\psi_0 = \star_0 \varphi_0$ where $\star_0$ is the Hodge star induced from g_0 and μ_0.

Definition

The group G_2 is the subgroup of $GL(\mathbb{R}, 7)$ that preserves the standard G_2 structure.

The Group G_2

It can be shown that g_0, φ_0, and μ_0 are related by

$$(a \lrcorner \varphi_0) \wedge (b \lrcorner \varphi_0) \wedge \varphi_0 = -6 g_0(a,b)\mu_0.$$

From this, we get that φ_0 determines g_0 and μ_0 in a non-linear way and that the group G_2

$$G_2 = \{A \in GL(\mathbb{R},7) \mid A^*\varphi_0 = \varphi_0\}.$$

G_2 Structures on Manifolds

Definition

A 3-form φ on a 7-manifold M is called a G_2 structure if for each $p \in M$ and $0 \neq Y_p \in T_pM$,

$$(Y_p \lrcorner \varphi_p) \wedge (Y_p \lrcorner \varphi_p) \wedge \varphi_p \neq 0.$$

Such a 3-form induces a metric g_7 and a Riemannian volume form vol_7 by the relation

$$(Y \lrcorner \varphi_0) \wedge (Z \lrcorner \varphi_0) \wedge \varphi_0 = -6g_7(a, b)\mathrm{vol}_7.$$

In turn, we obtain a Hodge star $\star_7$ and dual 4-form $\psi = \star_7\varphi$.

A 7-manifold M admits a G_2 structure if and only if it is spinnable and orientable.

Torsion

The spaces of forms on M can be decomposed into irreducible G_2 representations which will allow us to define the torsion forms.

Definition

Let φ be a G_2 structure. There are unique forms τ_0, τ_1, τ_2, and τ_3 called the torsion forms such that

$$d\varphi = \tau_0\psi + 3\tau_1 \wedge \varphi + \star\tau_3,$$

$$d\psi = 4\tau_1 \wedge \psi + \star\tau_2.$$

Types of G_2 Structure

The torsion forms allow us to define 16 classes of G_2 structure depending on which of the torsion forms are zero or non-zero.

Definition

A G_2 structure φ is:

- closed, if $d\varphi = 0$,
- coclosed, if $d\psi = 0$,
- torsion-free if it is both closed and coclosed.

When φ is torsion-free, the Riemannian holonomy of the metric g_7 is contained in the group G_2 and the metric g_7 is Ricci-flat (Fernández–Gray).

Laplacian Flow

Definition

A time-dependent G_2 structure φ_t defined on some interval $[0, T]$ satisfies the Laplacian flow equation if

$$\frac{\partial}{\partial t}\varphi_t = \Delta_{d_t}\varphi_t.$$

This flow was introduced by Bryant.

The critical points of the Laplacian flow are torsion-free G_2 structures.

Generally, we restrict our attention to closed G_2 structures when studying the Laplacian flow since in that case the closed condition is preserved.

Theorem (Bryant–Xu)

Let φ be a closed G_2 structure on a compact 7-fold M. Then, the Laplacian flow with initial condition φ has a unique solution for a short-time $[0, T]$ with T depending on φ.

Laplacian Coflow

Definition

A time-dependent G_2 structure φ_t on M defined on some interval $[0, T]$ satisfies the Laplacian coflow equation if

$$\frac{\partial}{\partial t}\psi_t = \Delta_{d_t}\psi_t.$$

This flow was first introduced by Karigiannis–McKay–Tsui (albeit with a minus sign).

The critical points of the Laplacian coflow are torsion-free G_2 structures.

Analogous to the Laplacian flow, we generally restrict our attention to coclosed G_2 structures.

Unlike the Laplacian flow, short-time existence and uniqueness for the Laplacian coflow is not known.

G_2 Structures from Calabi–Yau 2-Folds

Let (X^4, ω, Υ) be a Calabi–Yau 2-fold and choose:

- a nowhere-vanishing complex function F on X^4,
- and a strictly positive function G on X^4.

We can define a G_2 structure φ on $M^7 = T^3 \times X^4$ by setting

$$\varphi = -G dr^1 \wedge dr^2 \wedge dr^3 + dr^1 \wedge G\omega + dr^2 \wedge Re\left(\frac{F}{|\Upsilon|_\omega}\Upsilon\right) + dr^3 \wedge Im\left(\frac{F}{|\Upsilon|_\omega}\Upsilon\right).$$

Here r^1, r^2, and r^3 denote the angle coordinates on T^3.

G_2 Structures from Calabi–Yau 2-Folds

The associated metric and volume form are

$$g_7 = 2^{\frac{4}{3}}|F|^{-\frac{4}{3}}G^2(dr^1)^2 + 2^{-\frac{2}{3}}|F|^{\frac{2}{3}}(dr^2)^2 + 2^{-\frac{2}{3}}|F|^{\frac{2}{3}}(dr^3)^2 + 2^{-\frac{2}{3}}|F|^{\frac{2}{3}}g_4,$$

and

$$\mathrm{vol}_7 = 2^{-\frac{4}{3}}|F|^{\frac{4}{3}}G dr^1 \wedge dr^2 \wedge dr^3 \wedge \mathrm{vol}_4.$$

Lastly, we can compute the Hodge star and check that the dual 4-form ψ is

$$\psi = -2^{-\frac{4}{3}}|F|^{\frac{4}{3}} \cdot \frac{1}{2}\omega^2 + 2^{-\frac{4}{3}}|F|^{\frac{4}{3}} dr^2 \wedge dr^3 \wedge \omega + 2^{\frac{2}{3}}|F|^{-\frac{2}{3}} G dr^3 \wedge dr^1 \wedge Re\left(\frac{F}{|\Upsilon|_\omega}\Upsilon\right) + 2^{\frac{2}{3}}|F|^{-\frac{2}{3}} G dr^1 \wedge dr^2 \wedge Im\left(\frac{F}{|\Upsilon|_\omega}\Upsilon\right).$$

G_2 Structures from Calabi–Yau 3-Folds

We can do a similar thing on Calabi–Yau 3-folds. Let (X^6, ω, Υ) be a Calabi–Yau 3-fold and choose:

- a nowhere-vanishing complex function F on X^6,
- and a strictly positive function G on X^6.

We can define a G_2 structure φ on $M^7 = S^1 \times X^6$ by setting

$$\varphi = Re\left(\frac{F}{|\Upsilon|_\omega}\Upsilon\right) - dr \wedge G\omega,$$

where r is the angle coordinate on S^1.

G_2 Structures from Calabi–Yau 3-Folds

The associated metric and volume form are

$$g_7 = 4|F|^{-\frac{4}{3}} G^2 (dr)^2 + \frac{1}{2}|F|^{\frac{2}{3}} g_6,$$

and

$$\mathrm{vol}_7 = \frac{1}{4}|F|^{\frac{4}{3}} G dr^1 \wedge dr^2 \wedge dr^3 \wedge \mathrm{vol}_6.$$

We can again compute the Hodge star check that the dual 4-form ψ is

$$\psi = -2|F|^{-\frac{2}{3}} G dr \wedge Im\left(\frac{F}{|\Upsilon|_\omega}\Upsilon\right) - \frac{1}{4}|F|^{\frac{4}{3}} \cdot \frac{1}{2}\omega^2.$$

Closed G_2 Structures

With an appropriate choice of the functions F and G, we can obtain closed G_2 structures on the product manifolds. In particular $F = |\Upsilon|_\omega$ and $G = 1$ yields the forms

$$\varphi = -dr^1 \wedge dr^2 \wedge dr^3 + dr^1 \wedge \omega + dr^2 \wedge Re\,(\Upsilon) + dr^3 \wedge Im\,(\Upsilon) \text{ on } T^3 \times X^4,$$

and

$$\varphi = Re(\Upsilon) - dr \wedge \omega \text{ on } S^1 \times X^6.$$

Computing the Hodge Laplacians, we get

$$\Delta_d \varphi = 2^{\frac{2}{3}} \cdot \mathcal{L}_{\nabla_{(g_4)}\left(|\Upsilon|_\omega^{-\frac{2}{3}}\right)} \left(2dr^1 \wedge \omega - dr^2 \wedge Re\,(\Upsilon) - dr^3 \wedge Im\,(\Upsilon)\right) \text{ on } T^3 \times X^4,$$

and

$$\Delta_d \varphi = 2 \cdot \mathcal{L}_{\nabla_{(g_6)}\left(|\Upsilon|_\omega^{-\frac{2}{3}}\right)} \left(-Re\,(\Upsilon) - 2dr \wedge \omega\right) \text{ on } S^1 \times X^6.$$

The Torsion Forms

In both cases, we can compute the respect torsion forms of the G_2 structures. In particular, we have that

$$\tau_0 = 0, \qquad \tau_1 = 0, \qquad \tau_3 = 0,$$
$$\tau_2 = 2^{\frac{2}{3}} \cdot \left(\nabla_{(g_4)}(|\Upsilon|_\omega^{-\frac{2}{3}})\right) \lrcorner \left[-2dr^1 \wedge \omega + dr^2 \wedge Re\,(\Upsilon) + dr^3 \wedge Im\,(\Upsilon)\right] \text{ on } T^3 \times X^4,$$

and

$$\tau_0 = 0, \qquad \tau_1 = 0, \qquad \tau_3 = 0,$$
$$\tau_2 = 2 \cdot \left(\nabla_{(g_6)}(|\Upsilon|_\omega^{-\frac{2}{3}})\right) \lrcorner \left[Re\,(\Upsilon) + 2dr \wedge \omega\right] \text{ on } S^1 \times X^6.$$

The torsion forms vanish if and only if $|\Upsilon|_\omega$ is constant or equivalently when ω is Calabi–Yau.

Evolution Equations

If we assume that the Laplacian flow preserves the ansatz, then we have the evolution equation

$$\frac{\partial}{\partial t}\Big(- dr^1 \wedge dr^2 \wedge dr^3 + dr^1 \wedge \omega_t + dr^2 \wedge Re\,(\Upsilon_t) + dr^3 \wedge Im\,(\Upsilon_t)\Big)$$
$$= 2^{\frac{2}{3}} \cdot \mathcal{L}_{\nabla_{(g_4)_t}\left(|\Upsilon_t|_{\omega_t}^{-\frac{2}{3}}\right)}\Big(2dr^1 \wedge \omega_t - dr^2 \wedge Re\,(\Upsilon_t) - dr^3 \wedge Im\,(\Upsilon_t)\Big) \text{ on } T^3 \times X^4,$$

and

$$\frac{\partial}{\partial t}\Big(Re(\Upsilon_t) - dr \wedge \omega_t\Big) = 2 \cdot \mathcal{L}_{\nabla_{(g_6)_t}\left(|\Upsilon_t|_{\omega_t}^{-\frac{2}{3}}\right)}\Big(- Re\,(\Upsilon_t) - 2dr \wedge \omega_t\Big) \text{ on } S^1 \times X^6.$$

Idea

The angle coordinates are not affected by the Lie derivatives or time derivatives. The terms involving ω_t and Υ_t are similar in both cases so we can tackle them simultaneously.

Evolution Equations

If we assume that the Laplacian flow preserves the ansatz, then we have the evolution equation

$$\frac{\partial}{\partial t}\Big(- dr^1 \wedge dr^2 \wedge dr^3 + dr^1 \wedge \omega_t + dr^2 \wedge Re\left(\Upsilon_t\right) + dr^3 \wedge Im\left(\Upsilon_t\right)\Big)$$
$$= 2^{\frac{2}{3}} \cdot \mathcal{L}_{\nabla_{(g_4)_t}\left(|\Upsilon_t|_{\omega_t}^{-\frac{2}{3}}\right)}\Big(2dr^1 \wedge \omega_t - dr^2 \wedge Re\left(\Upsilon_t\right) - dr^3 \wedge Im\left(\Upsilon_t\right)\Big) \text{ on } T^3 \times X^4,$$

and

$$\frac{\partial}{\partial t}\Big(Re(\Upsilon_t) - dr \wedge \omega_t\Big) = 2 \cdot \mathcal{L}_{\nabla_{(g_6)_t}\left(|\Upsilon_t|_{\omega_t}^{-\frac{2}{3}}\right)}\Big(- Re\left(\Upsilon_t\right) - 2dr \wedge \omega_t\Big) \text{ on } S^1 \times X^6.$$

Idea

The angle coordinates are not affected by the Lie derivatives or time derivatives. The terms involving ω_t and Υ_t are similar in both cases so we can tackle them simultaneously.

Evolution Equations

Let h_t denote the metric in either case.

Matching the ω_t and Υ_t terms with each other, we are left with the following evolution equations:

$$\frac{\partial}{\partial t}\omega_t = 2K \cdot \mathcal{L}_{\nabla_{h_t}\left(|\Upsilon_t|_{\omega_t}^{-\frac{2}{3}}\right)}\omega_t,$$

$$\frac{\partial}{\partial t}\Upsilon_t = -K \cdot \mathcal{L}_{\nabla_{h_t}\left(|\Upsilon_t|_{\omega_t}^{-\frac{2}{3}}\right)}\Upsilon_t,$$

with $K = 2^{\frac{n}{3}}$ being a constant depending only on the dimension of the base manifold.

Calabi–Yau structures satisfying the above equations will induce G_2 structures that satisfy the Laplacian flow.

Remark

A priori, it is not clear that structures satisfying the above evolution equations remain compatible as Calabi–Yau structures.

Evolution Equations

We can expand the Lie derivative terms in the evolution equations. Working on the first equation, we get

$$\frac{\partial}{\partial t}\omega_t = 2K \cdot \mathcal{L}_{\nabla_{h_t}\left(|\Upsilon_t|_{\omega_t}^{-\frac{2}{3}}\right)}\omega_t = 4K \cdot i\partial_t\overline{\partial}_t(|\Upsilon_t|_{\omega_t}^{-\frac{2}{3}}).$$

This equation will be related to the MA$^{\frac{1}{3}}$ flow.

The Lie derivative term in the second equation

$$\frac{\partial}{\partial t}\Upsilon_t = -K \cdot \mathcal{L}_{\nabla_{h_t}\left(|\Upsilon_t|_{\omega_t}^{-\frac{2}{3}}\right)}\Upsilon_t,$$

is in general not an $(n, 0)$-form with respect to the complex structure J_t. This implies that the complex structure must also evolve in time to preserve compatibility.

Idea

In order to address the compatibility conditions (and the fact that J_t needs to change), we can look for solutions by acting on compatible Calabi–Yau structures via a moving family of diffeomorphisms. This idea is similar to that of Fei–Phong–Picard–Zhang.

A Solution from the MA$^{\frac{1}{3}}$ Flow

Fix an initial Calabi–Yau structure (ω, Υ) on a compact Calabi–Yau n-fold X.

The MA$^{\frac{1}{3}}$ flow then gives the existence of a solution u_t to the equation

$$\frac{\partial}{\partial t}u_t = 6K \cdot \left(e^{-2\log|\Upsilon|_\omega}\frac{\det(\omega + i\partial\overline{\partial}u_t)}{\det\omega}\right)^{\frac{1}{3}}, \qquad \omega + i\partial\overline{\partial}u_t > 0, \qquad u_0 = 0.$$

In turn, we get a family of Kähler metrics $\widetilde{\omega}_t = \omega + i\partial\overline{\partial}u_t$ which converge to the Calabi–Yau metric ω_{CY} that also satisfy

$$\frac{\partial}{\partial t}\widetilde{\omega}_t = 6K \cdot i\partial\overline{\partial}\frac{\partial}{\partial t}u_t = 6K \cdot i\partial\overline{\partial}(|\Upsilon|_{\widetilde{\omega}_t})^{-\frac{2}{3}}.$$

A Solution from the MA$^{\frac{1}{3}}$ Flow

Using the time-dependent Kähler metrics, we can define a vector field Y by

$$Y_t = -K \cdot \nabla_{\widetilde{h}_t}(|\Upsilon|_{\widetilde{\omega}_t}^{-\frac{2}{3}}).$$

The vector field Y determines a 1-parameter family of diffeomorphisms such that

$$\frac{\partial}{\partial t}\Theta_t(p) = Y_t(\Theta_t(p)), \qquad \Theta_0 = \mathrm{id}_X.$$

Using Θ_t, we can pull our tensors back. The pullback structures

$$\omega_t = \Theta_t^*\widetilde{\omega}_t, \qquad \Upsilon_t = \Theta_t^*\Upsilon, \qquad J_t = \Theta_t^*J, \qquad h_t = \Theta_t^*\widetilde{h}_t,$$

remain compatible with one another as Calabi–Yau structures.

A Solution from the MA$^{\frac{1}{3}}$ Flow

Using DeTurck's trick, we can show that ω_t and Υ_t satisfy our desired evolution equations.

$$
\begin{aligned}
\frac{\partial}{\partial t}\omega_t &= \frac{\partial}{\partial t}(\Theta_t^*\widetilde{\omega}_t) = \Theta_t^*(\mathcal{L}_{Y_t}\widetilde{\omega}_t) + \Theta_t^*\left(\frac{\partial}{\partial t}\widetilde{\omega}_t\right) \\
&= \mathcal{L}_{(\Theta_t^{-1})_* Y_t}(\Theta_t^*\widetilde{\omega}_t) + \Theta_t^*\left(6K \cdot i\partial\overline{\partial}(|\Upsilon|_{\widetilde{\omega}_t}^{-\frac{2}{3}})\right) \\
&= \mathcal{L}_{-K\cdot(\Theta_t^{-1})_*[\nabla_{\widetilde{h}_t}(|\Upsilon|_{\widetilde{\omega}_t}^{-\frac{2}{3}})]}\omega_t + 6K \cdot i\partial_t\overline{\partial}_t\left(\Theta_t^*(|\Upsilon|_{\widetilde{\omega}_t}^{-\frac{2}{3}})\right) \\
&= -K \cdot \mathcal{L}_{\nabla_{h_t}(|\Upsilon_t|_{\omega_t}^{-\frac{2}{3}})}\omega_t + 6K \cdot i\partial_t\overline{\partial}_t(|\Upsilon_t|_{\omega_t}^{-\frac{2}{3}}) = 2K \cdot \mathcal{L}_{\nabla_{h_t}(|\Upsilon_t|_{\omega_t}^{-\frac{2}{3}})}\omega_t.
\end{aligned}
$$

Similarly, we can check that

$$
\begin{aligned}
\frac{\partial}{\partial t}\Upsilon_t &= \frac{\partial}{\partial t}(\Theta_t^*\Upsilon) = \Theta_t^*(\mathcal{L}_{Y_t}\Upsilon) \\
&= \mathcal{L}_{(\Theta_t^{-1})_* Y_t}(\Theta_t^*\Upsilon) = -K \cdot \mathcal{L}_{\nabla_{h_t}(|\Upsilon_t|_{\omega_t}^{-\frac{2}{3}})}\Upsilon_t,
\end{aligned}
$$

Summary

The structures (ω_t, Υ_t) satisfy the desired evolution equations

$$\frac{\partial}{\partial t}\omega_t = 2K \cdot \mathcal{L}_{\nabla_{h_t}(|\Upsilon_t|_{\omega_t}^{-\frac{2}{3}})}\omega_t$$

$$\frac{\partial}{\partial t}\Upsilon_t = -K \cdot \mathcal{L}_{\nabla_{h_t}(|\Upsilon_t|_{\omega_t}^{-\frac{2}{3}})}\Upsilon_t.$$

They are compatible as Calabi–Yau structures since they are obtained as pullbacks of compatible structures.

Thus, their associated G_2 structures

$$\varphi = -dr^1 \wedge dr^2 \wedge dr^3 + dr^1 \wedge \omega_t + dr^2 \wedge Re(\Upsilon_t) + dr^3 \wedge Im(\Upsilon_t) \text{ on } T^3 \times X^4,$$

and

$$\varphi = Re(\Upsilon_t) - dr \wedge \omega_t \text{ on } S^1 \times X^6.$$

satisfy the Laplacian flow equation.

Uniqueness of the Laplacian flow tells us that this solution is unique given the initial condition.

Coclosed G_2 Structures

Reversing our choices for F and G, we can obtain coclosed G_2 structures on the product manifolds.

$$\psi = -2^{-\frac{4}{3}} \cdot \frac{1}{2}\omega^2 + 2^{-\frac{4}{3}} dr^2 \wedge dr^3 \wedge \omega + 2^{\frac{2}{3}} dr^3 \wedge dr^1 \wedge Re\,(\Upsilon) + 2^{\frac{2}{3}} dr^1 \wedge dr^2 \wedge Im\,(\Upsilon) \text{ on } T^3 \times X^4,$$

and

$$\psi = -2dr \wedge Im\,(\Upsilon) - \frac{1}{4} \cdot \frac{1}{2}\omega^2 \text{ on } S^1 \times X^6.$$

Computing the Hodge Laplacians, we get

$$\Delta_d \psi = 2^{\frac{2}{3}} \cdot \mathcal{L}_{\nabla_{(g_4)}(\log|\Upsilon|_\omega)} \Big(2^{-\frac{4}{3}} \cdot \frac{1}{2}\omega^2 - 2^{-\frac{4}{3}} dr^2 \wedge dr^3 \wedge \omega + 2^{\frac{2}{3}} dr^3 \wedge dr^1 \wedge Re\,(\Upsilon) + 2^{\frac{2}{3}} dr^1 \wedge dr^2 \wedge Im\,(\Upsilon)\Big) \text{ on } T^3 \times X^4,$$

and

$$\Delta_d \psi = 2 \cdot \mathcal{L}_{\nabla_{(g_6)}(\log|\Upsilon|_\omega)} \Big(-2dr \wedge Im\,(\Upsilon) + \frac{1}{4} \cdot \frac{1}{2}\omega^2 \Big) \text{ on } S^1 \times X^6.$$

The Torsion Forms

In both cases, we can compute the respect torsion forms of the G_2 structures. In particular, we have that

$$\tau_0 = 0, \qquad \tau_1 = 0, \qquad \tau_2 = 0,$$
$$\tau_3 = 2^{\frac{2}{3}} \cdot \left(\nabla_{(g_4)}(\log|\Upsilon|_\omega)\right) \lrcorner \Big[2^{-\frac{4}{3}} \cdot \frac{1}{2}\omega^2 - 2^{-\frac{4}{3}} dr^2 \wedge dr^3 \wedge \omega + 2^{\frac{2}{3}} dr^3 \wedge dr^1 \wedge Re\,(\Upsilon) + 2^{\frac{2}{3}} dr^1 \wedge dr^2 \wedge Im\,(\Upsilon)\Big] \text{ on } T^3 \times X^4,$$

and

$$\tau_0 = 0, \qquad \tau_1 = 0, \qquad \tau_2 = 0,$$
$$\tau_3 = 2 \cdot \left(\nabla_{(g_6)}(\log|\Upsilon|_\omega)\right) \lrcorner \left[-2dr \wedge Im\,(\Upsilon) + \frac{1}{4} \cdot \frac{1}{2}\omega^2\right] \text{ on } S^1 \times X^6.$$

The torsion forms vanish if and only if $|\Upsilon|_\omega$ is constant or equivalently when ω is Calabi–Yau.

Evolution Equations

If we assume that the Laplacian coflow preserves the ansatz, then we have the evolution equation

$$\frac{\partial}{\partial t}\Big(-2^{-\frac{4}{3}}\cdot\frac{1}{2}\omega_t^2+2^{-\frac{4}{3}}dr^2\wedge dr^3\wedge\omega_t + 2^{\frac{2}{3}}dr^3\wedge dr^1\wedge Re\left(\Upsilon_t\right)+2^{\frac{2}{3}}dr^1\wedge dr^2\wedge Im\left(\Upsilon_t\right)\Big)$$
$$=2^{\frac{2}{3}}\cdot\mathcal{L}_{\nabla_{(g_4)_t}(\log|\Upsilon_t|_{\omega_t})}\Big(2^{-\frac{4}{3}}\cdot\frac{1}{2}\omega_t^2-2^{-\frac{4}{3}}dr^2\wedge dr^3\wedge\omega_t + 2^{\frac{2}{3}}dr^3\wedge dr^1\wedge Re\left(\Upsilon_t\right)+2^{\frac{2}{3}}dr^1\wedge dr^2\wedge Im\left(\Upsilon_t\right)\Big) \text{ on } T^3\times X^4,$$

and

$$\frac{\partial}{\partial t}\Big(-2dr\wedge Im\left(\Upsilon_t\right)-\frac{1}{4}\cdot\frac{1}{2}\omega_t^2\Big)$$
$$=2\cdot\mathcal{L}_{\nabla_{(g_6)_t}(\log|\Upsilon_t|_{\omega_t})}\Big(-2dr\wedge Im\left(\Upsilon_t\right)+\frac{1}{4}\cdot\frac{1}{2}\omega_t^2\Big) \text{ on } S^1\times X^6.$$

Evolution Equations

Under the assumption that the ansatz is preserved, we have the evolution equation

$$\frac{\partial}{\partial t}\Big(-2^{-\frac{4}{3}} \cdot \frac{1}{2}\omega_t^2 + 2^{-\frac{4}{3}} dr^2 \wedge dr^3 \wedge \omega_t$$
$$+ 2^{\frac{2}{3}} dr^3 \wedge dr^1 \wedge Re\,(\Upsilon_t) + 2^{\frac{2}{3}} dr^1 \wedge dr^2 \wedge Im\,(\Upsilon_t)\Big)$$
$$= 2^{\frac{2}{3}} \cdot \mathcal{L}_{\nabla_{(g_4)_t}(\log|\Upsilon_t|_{\omega_t})}\Big(2^{-\frac{4}{3}} \cdot \frac{1}{2}\omega_t^2 - 2^{-\frac{4}{3}} dr^2 \wedge dr^3 \wedge \omega_t$$
$$+ 2^{\frac{2}{3}} dr^3 \wedge dr^1 \wedge Re\,(\Upsilon_t) + 2^{\frac{2}{3}} dr^1 \wedge dr^2 \wedge Im\,(\Upsilon_t)\Big) \text{ on } T^3 \times X^4,$$

and

$$\frac{\partial}{\partial t}\Big(-2dr \wedge Im\,(\Upsilon_t) - \frac{1}{4} \cdot \frac{1}{2}\omega_t^2\Big)$$
$$= 2 \cdot \mathcal{L}_{\nabla_{(g_6)_t}(\log|\Upsilon_t|_{\omega_t})}\Big(-2dr \wedge Im\,(\Upsilon_t) + \frac{1}{4} \cdot \frac{1}{2}\omega_t^2\Big) \text{ on } S^1 \times X^6.$$

Evolution Equations

Matching terms again, we get:

$$\frac{\partial}{\partial t}\omega_t = -K \cdot \mathcal{L}_{\nabla_{h_t}(\log|\Upsilon_t|_{\omega_t})}\omega_t,$$

$$\frac{\partial}{\partial t}\Upsilon_t = K \cdot \mathcal{L}_{\nabla_{h_t}(\log|\Upsilon_t|_{\omega_t})}\Upsilon_t,$$

with $K = 2^{\frac{n}{3}}$.

Calabi–Yau structures satisfying the above equations will induce G_2 structures that satisfy the Laplacian coflow.

Remark

As before, it is not clear that structures satisfying the above evolution equations remain compatible as Calabi–Yau structures.

Evolution Equations

Working with the Lie derivative terms in the evolution equations, we get

$$\frac{\partial}{\partial t}\omega_t = -K \cdot \mathcal{L}_{\nabla_{h_t}(\log|\Upsilon_t|_{\omega_t})}\omega_t = -2K \cdot i\partial_t\overline{\partial}_t(\log|\Upsilon_t|_{\omega_t}) = -K \cdot \mathrm{Ric}(\omega_t, J_t).$$

This is reminiscent of the Kähler–Ricci flow.

In the second equation,

$$\frac{\partial}{\partial t}\Upsilon_t = K \cdot \mathcal{L}_{\nabla_{h_t}(\log|\Upsilon_t|_{\omega_t})}\Upsilon_t,$$

is again in general not an $(n, 0)$-form and so the complex structure J_t has to change with time.

Idea

We can again look for solutions by acting on compatible Calabi–Yau structures via a moving family of diffeomorphisms.

A Solution from the Kähler–Ricci Flow

Fix an initial Calabi–Yau structure (ω, Υ) on a compact Calabi–Yau n-fold X.

The (rescaled) Kähler–Ricci flow then gives the existence of a family of Kähler metrics $\widetilde{\omega}_t$ which converge to the Calabi–Yau metric ω_{CY} that also satisfy

$$\frac{\partial}{\partial t}\widetilde{\omega}_t = -2K \cdot \mathrm{Ric}(\widetilde{\omega}_t, J), \qquad \widetilde{\omega}_0 = \omega.$$

The Kähler metrics define a time-dependent vector field Y by

$$Y_t = K \cdot \nabla_{\widetilde{h}_t}(\log |\Upsilon|_{\widetilde{\omega}_t}).$$

We in turn obtain a 1-parameter family of diffeomorphisms such that

$$\frac{\partial}{\partial t}\Theta_t(p) = Y_t(\Theta_t(p)), \qquad \Theta_0 = \mathrm{id}_X.$$

We then define the pullback structures

$$\omega_t = \Theta_t^*\widetilde{\omega}_t, \qquad \Upsilon_t = \Theta_t^*\Upsilon, \qquad J_t = \Theta_t^*J, \qquad h_t = \Theta_t^*\widetilde{h}_t.$$

A Solution from the Kähler–Ricci Flow

Using DeTurck's trick, we can show that ω_t and Υ_t satisfy our desired evolution equations.

$$\begin{aligned}\frac{\partial}{\partial t}\omega_t &= \frac{\partial}{\partial t}(\Theta_t^*\widetilde{\omega}_t) = \Theta_t^*(\mathcal{L}_{Y_t}\widetilde{\omega}_t) + \Theta_t^*\Big(\frac{\partial}{\partial t}\widetilde{\omega}_t\Big)\\ &= \mathcal{L}_{(\Theta_t^{-1})_* Y_t}(\Theta_t^*\widetilde{\omega}_t) + \Theta_t^*\Big(-2K\cdot \mathrm{Ric}(\widetilde{\omega}_t, J)\Big)\\ &= \mathcal{L}_{K\cdot(\Theta_t^{-1})_*[\nabla_{\widetilde{h}_t}(\log|\Upsilon|_{\widetilde{\omega}_t})]}\omega_t - 2K\cdot\mathrm{Ric}(\Theta_t^*\widetilde{\omega}_t, \Theta_t^* J)\\ &= K\cdot\mathcal{L}_{\nabla_{h_t}(\log|\Upsilon_t|_{\omega_t})}\omega_t - 2K\cdot\mathrm{Ric}(\omega_t, J_t) = -K\cdot\mathcal{L}_{\nabla_{h_t}(\log|\Upsilon_t|_{\omega_t})}\omega_t.\end{aligned}$$

Similarly, we can check that

$$\begin{aligned}\frac{\partial}{\partial t}\Upsilon_t &= \frac{\partial}{\partial t}(\Theta_t^*\Upsilon) = \Theta_t^*(\mathcal{L}_{Y_t}\Upsilon)\\ &= \mathcal{L}_{(\Theta_t^{-1})_* Y_t}(\Theta_t^*\Upsilon) = K\cdot\mathcal{L}_{\nabla_{h_t}(\log|\Upsilon_t|_{\omega_t})}\Upsilon_t,\end{aligned}$$

Summary

The structures (ω_t, Υ_t) satisfy the desired evolution equations

$$\frac{\partial}{\partial t}\omega_t = -K \cdot \mathcal{L}_{\nabla_{h_t}(\log|\Upsilon_t|_{\omega_t})}\omega_t$$

$$\frac{\partial}{\partial t}\Upsilon_t = K \cdot \mathcal{L}_{\nabla_{h_t}(\log|\Upsilon_t|_{\omega_t})}\Upsilon_t.$$

They are compatible as Calabi–Yau structures since they are obtained as pullbacks of compatible structures.

It follows that the associated G_2 structures

$$\psi = -2^{-\frac{4}{3}} \cdot \frac{1}{2}\omega^2 + 2^{-\frac{4}{3}} dr^2 \wedge dr^3 \wedge \omega + 2^{\frac{2}{3}} dr^3 \wedge dr^1 \wedge Re(\Upsilon) + 2^{\frac{2}{3}} dr^1 \wedge dr^2 \wedge Im(\Upsilon) \text{ on } T^3 \times X^4,$$

and

$$\psi = -2dr \wedge Im(\Upsilon) - \frac{1}{4} \cdot \frac{1}{2}\omega^2 \text{ on } S^1 \times X^6$$

satisfy the Laplacian coflow equation.

The Story So Far

We have found a family of solutions to the Laplacian flow and coflow in terms of Calabi–Yau structures on the base manifold.

In all cases, we have:

- The flow is solved by a pair $(\omega_t, \Upsilon_t) = (\Theta_t^* \widetilde{\omega}_t, \Theta_t^* \Upsilon)$,
- The time-dependent family of Kähler triples $(\widetilde{\omega}_t, J, \widetilde{h}_t)$ come from a Monge–Ampère flow,
- The Kähler metrics $\widetilde{\omega}_t$ satisfy uniform estimates and exponential convergence conditions. They also converge to the Calabi–Yau metric ω_{CY} in the Kähler class $[\omega]$ in each $C^k(X, h)$-norm,
- The diffeomorphisms Θ_t solve $\frac{\partial}{\partial t}\Theta_t = Y_t$, where Y_t is a time-dependent vector field defined using derivatives of (powers and logarithms of) the norm $|\Upsilon|_{\widetilde{\omega}_t}$.

With these ingredients, we will prove convergence of the structures (ω_t, Υ_t) and their associated G_2 structures (borrowing ideas from Lotay–Wei).

The Limit Diffeomorphism

Recall that Y_t was defined either by

$$Y_t = -K \cdot \nabla_{\widetilde{h}_t}(|\Upsilon|_{\widetilde{\omega}_t}|^{-\frac{2}{3}}) \text{ or } Y_t = K \cdot \nabla_{\widetilde{h}_t}(\log|\Upsilon|_{\widetilde{\omega}_t}|),$$

and that the Calabi–Yau metric ω_{CY} has the property that the norm $|\Upsilon|_{\omega_{CY}}$ is constant.

It follows that the vector field Y_t converges to 0 exponentially fast in each $C^k(X, h)$ norm and so there exist positive constant C_k, λ_k such that

$$|\nabla_h^k Y_t|_h \leq C_k e^{-\lambda_k t}.$$

Given a point $p \in X$, and $t_1, t_2 \geq 0$, we can define a smooth path γ from $\Theta_{t_1}(p)$ to $\Theta_{t_2}(p)$ by

$$\gamma(t) = \Theta_t(p).$$

We then see that

$$d_h(\Theta_{t_1}(p), \Theta_{t_2}(p)) \leq \int_{t_1}^{t_2} \left|\frac{\partial}{\partial t}\Theta_t(p)\right|_h dt \leq \int_{t_1}^{t_2} |Y_t|_h dt \leq C_0 \int_{t_1}^{t_2} e^{-\lambda_0 t} dt,$$

and so the maps Θ_t converge uniformly with respect to h.

The other uniform estimates show that the Θ_t converge in each $C^k(X, h)$-norm and so we have some limit map Θ_∞.

The Limit Diffeomorphism

Next, for each t, we have

$$\left|\frac{\partial}{\partial t}\log\left(\frac{\Upsilon_t\wedge\overline{\Upsilon}_t}{\Upsilon\wedge\overline{\Upsilon}}\right)\right| = \left|\frac{\partial}{\partial t}\left(\log\frac{\Theta_t^*(\Upsilon\wedge\overline{\Upsilon})}{\Upsilon\wedge\overline{\Upsilon}}\right)\right| = \left|\frac{1}{\Theta_t^*(\Upsilon\wedge\overline{\Upsilon})}\frac{\partial}{\partial t}\left(\Theta_t^*(\Upsilon\wedge\overline{\Upsilon})\right)\right|$$
$$= \left|\Theta_t^*\left(\frac{\mathcal{L}_{Y_t}(\Upsilon\wedge\overline{\Upsilon})}{\Upsilon\wedge\overline{\Upsilon}}\right)\right| \leq \sup_X\left|\left(\frac{\mathcal{L}_{Y_t}(|\Upsilon|^2_\omega \mathrm{vol})}{|\Upsilon|^2_\omega \mathrm{vol}}\right)\right|$$
$$\leq \frac{|Y_t(|\Upsilon|^2_\omega)|}{|\Upsilon|^2_\omega} + \left|\frac{d(Y_t \lrcorner \mathrm{vol})}{\mathrm{vol}}\right| \leq Ce^{-\lambda t}.$$

It follows that

$$\left|\log\left(\frac{\Upsilon_t\wedge\overline{\Upsilon}_t}{\Upsilon\wedge\overline{\Upsilon}}\right)\right| \leq \int_0^t\left|\frac{\partial}{\partial s}\log\left(\frac{\Upsilon_s\wedge\overline{\Upsilon}_s}{\Upsilon\wedge\overline{\Upsilon}}\right)\right|ds \leq \int_0^t e^{-\lambda s}ds \leq C$$

is uniformly bounded.

This gives another uniform estimate

$$C^{-1}\cdot(\Upsilon\wedge\overline{\Upsilon}) \leq \Theta_t^*(\Upsilon\wedge\overline{\Upsilon}) \leq C\cdot(\Upsilon\wedge\overline{\Upsilon}),$$

and so the pullbacks Θ_t^* are uniformly non-degenerate.

We get that $\det(\Theta_t^*)$ is uniformly bounded and this estimate can be passed to the limit Θ_∞.

The Limit Diffeomorphism

The map Θ_∞ is a local diffeomorphism by the Inverse Function Theorem.

Each Θ_t is also a diffeomorphism isotopic to the identity, and so Θ_∞ is surjective and homotopic to the identity.

Since X is compact, Θ_∞ is a covering map. As Θ_∞ is homotopic to the identity, it has degree 1 and is injective.

Thus Θ_∞ is a bijective local diffeomorphism and hence a diffeomorphism.

It follows that (ω_t, Υ_t) converge to $(\Theta_\infty^* \omega_{CY}, \Theta_\infty^* \Upsilon)$ as $t \to \infty$.

Limiting G_2 Structures

We can apply this to the Laplacian flow of the associated G_2 structures.

Theorem (Picard–S.)

Start the Laplacian flow with initial data

$$\varphi = -dr \wedge dr^2 \wedge dr^3 + dr^1 \wedge \omega + dr^2 \wedge Re\,(\Upsilon) + dr^3 \wedge Im\,(\Upsilon) \text{ on } T^3 \times X^4,$$

or

$$\varphi = Re\,(\Upsilon) - dr \wedge \omega \text{ or } S^1 \times X^6.$$

Then the Laplacian flow exists for all time t and is given by the $MA^{\frac{1}{3}}$ flow (up to diffeomorphism) and converges to a stationary point

$$\begin{aligned}\varphi_\infty = &-dr \wedge dr^2 \wedge dr^3 + dr^1 \wedge \Theta_\infty^* \omega_{CY} \\ &+ dr^2 \wedge Re\,(\Theta_\infty^* \Upsilon) + dr^3 \wedge Im\,(\Theta_\infty^* \Upsilon) \text{ on } T^3 \times X^4\end{aligned}$$

or

$$\varphi_\infty = Re\,(\Theta_\infty^* \Upsilon) - dr \wedge \Theta_\infty^* \omega_{CY} \text{ on } S^1 \times X^6,$$

where Θ_∞ is a diffeomorphism on the base and ω_{CY} is the unique Ricci-flat Kähler metric in the class $[\omega]$.

Limiting G_2 Structures

We have the analogous result for the Laplacian coflow.

Theorem (Picard–S.)

Start the Laplacian coflow with initial data

$$\psi = -2^{-\frac{4}{3}} \cdot \frac{1}{2}\omega^2 + 2^{-\frac{4}{3}} \cdot dr^2 \wedge dr^3 \wedge \omega + 2^{\frac{2}{3}} \cdot dr^3 \wedge dr^1 \wedge Re\,(\Upsilon) + 2^{\frac{2}{3}} \cdot dr^1 \wedge dr^2 \wedge Im\,(\Upsilon) \text{ on } T^3 \times X^4$$

or

$$\psi = -2 \cdot dr \wedge Im\,(\Upsilon) - \frac{1}{4} \cdot \frac{1}{2}\omega^2 \text{ on } S^1 \times X^6.$$

Then the Laplacian coflow exists for all time t and is given by the Kähler–Ricci flow (up to diffeomorphism) and converges to a stationary point

$$\psi_\infty = -2^{-\frac{4}{3}} \cdot \frac{1}{2}\Theta_\infty^*\omega_{CY}^2 + 2^{-\frac{4}{3}} \cdot dr^2 \wedge dr^3 \wedge \Theta_\infty^*\omega_{CY} + 2^{\frac{2}{3}} \cdot dr^3 \wedge dr^1 \wedge Re\,(\Theta_\infty^*\Upsilon) + 2^{\frac{2}{3}} \cdot dr^1 \wedge dr^2 \wedge Im\,(\Theta_\infty^*\Upsilon) \text{ on } T^3 \times X^4$$

or

$$\psi_\infty = -2 \cdot dr \wedge Im\,(\Theta_\infty^*\Upsilon) - \frac{1}{4} \cdot \frac{1}{2}\Theta_\infty^*\omega_{CY}^2 \text{ on } S^1 \times X^6,$$

where Θ_∞ is a diffeomorphism on the base and ω_{CY} is the unique Ricci-flat Kähler metric in the class $[\omega]$.

Thank you.