# Deformations of Special Structures in Dimensions 6 and 7

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# Abstract

In this thesis, we study three problems in non-Kähler Calabi–Yau geometry and $G_2$-geometry centered on Reid's Fantasy [Rei87] and the Hull–Strominger system [Hul86, Str86].

The first concerns the geometrization of conifold transitions, processes which allow us to traverse the moduli space of compact Calabi–Yau threefolds. Work of Fu–Li–Yau [FLY12] and of Collins–Picard–Yau [CPY24] has constructed metrics on both sides of this process which are partial solutions to the Hull–Strominger system. Using these (conformally) balanced and Hermitian Yang–Mills metrics, we show that conifold transitions are continuous in the Gromov–Hausdorff topology.

The next focuses on the Anomaly flow of Phong–Picard–Zhang [PPZ18b]. We extend their ideas from the $\alpha' = 0$ case and compute integral Shi-type estimates along the flow for general slope parameter $\alpha'$. We achieve this by adapting an integration-by-parts type argument instead of the usual Maximum Principle techniques in order to deal with the extra terms that appear. From this, we obtain a smallness condition on $\alpha'$ that allows the flow to be extended from $[0, \tau)$ to a larger interval $[0, \tau + \epsilon)$.

Finally, we study the relationship between Calabi–Yau geometry and $G_2$-geometry by considering geometric flows on $S^1$-fibrations over Calabi–Yau threefolds. In particular, we construct families of closed and coclosed $G_2$-structures on these fibrations and apply the Laplacian flow and (modified) coflow respectively. Using these Ansätze, we show that on a trivial fibration these flows reduce to particular Monge–Ampère flows on the base manifold. We perform a similar analysis on contact Calabi–Yau 7-folds and obtain conditions for these families to satisfy the flows.

# Lay Summary

This thesis studies several problems related to the Hull–Strominger system – a system of equations from heterotic string theory that provide supersymmetric conditions on a manifold.

First, we show that a certain type of deformation – called a conifold transition – is continuous when using special structures from the Hull–Strominger system, despite passing through an intermediate space with singularities.

Next, study the long-time behaviour of the Anomaly flow – a geometric method of finding solutions to the system. In particular, we prove a smallness condition on a parameter $\alpha'$ which allows us to extend the flow further.

Finally, we consider the relationship between SU(3)- and $G_2$-structures, which respectively come from special geometries in 6 and 7 dimensions. We do this by deforming $G_2$-structures and noting their effect on the underlying SU(3)-structures.

# Preface

This thesis is split into two parts and is based on four preprints written during the course of my doctoral studies. Part I covers topics in non-Kähler Calabi–Yau geometry and is largely based on joint work with Benjamin Friedman and Sébastien Picard [FPS24], as well as a singly-authored paper [Sua24]. Both of these works have been submitted for publication.

Part II covers topics related to flows in $G_2$-geometry and their relation to (transverse) Calabi–Yau geometry. It is based on a joint paper with Sébastien Picard [PS24] (published in Mathematical Research Letters 31 (2024) No. 6, pp. 1837 – 1877) and a preprint written with Henrique N. Sá Earp and Julieth Saavedra [SESS24] which has also been submitted for publication.

Work in each collaborative project was done equally by all contributing authors.

No generative AI was used for any aspects of this work.

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I am thankful for the opportunities that I had working on projects alongside Henrique Sá Earp, Julieth Saavedra, and Benji Friedman. It was an insightful experience to be able to bounce ideas around with them and learn new topics from different perspectives.

The people at the Department of Mathematics at UBC has been extremely helpful during my time here. I want to express my gratitude to Jingyi Chen and Albert Chau for being part of my supervisory committee and providing advice to me along the way. I additionally thank Fok Shuen Leung for his help regarding teaching in recent years. Despite me not being on campus particularly often, I also appreciate the friends and acquaintances that I have made along the way at UBC. It was always a pleasure to have someone to work with on assignments, chat with about math, or lament some of our shared experiences as graduate students.

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Last but not least, I thank God for His blessings and favour upon me.

# Dedication

To everyone who misread “Calabi” as “Caleb”.

# Part I

# Non-Kähler Calabi–Yau Geometry

# Chapter 1

# Introduction

We begin with an introduction on non-Kähler Calabi–Yau geometry, specializing quickly to the complex dimension 3 setting.

## 1.1 Calabi–Yau Manifolds

**Definition 1.1.1.** A Calabi–Yau $n$-fold $X$ is a compact complex manifold of complex dimension $n$ with finite fundamental group and trivial canonical bundle.

In general, we do not require that a Calabi–Yau manifold admit a Kähler metric and will explicitly state when one is assumed to exist.

Since $X$ has trivial canonical bundle, we have that there exists some nowhere-vanishing holomorphic $(n, 0)$-form $\Upsilon$ on $X$. Given a Hermitian metric $\omega$ on $X$, we can write the norm of $\Upsilon$ with respect to $\omega$ by the local formula

$$\|\Upsilon\|_\omega^2 = \frac{|f|^2}{\det g_{p\overline{q}}}. \tag{1.1}$$

Here

$$\omega = \sqrt{-1} g_{j\overline{k}} dz^j \wedge d\overline{z}^{\overline{k}} \tag{1.2}$$

and

$$\Upsilon = f dz^1 \wedge \ldots \wedge dz^n. \tag{1.3}$$

where $f$ is a local holomorphic function.

We will sometimes refer to a pair $(\omega, \Upsilon)$ as a SU(3)-structure.

If $\omega$ is a Ricci-flat Kähler metric, then the norm $\|\Upsilon\|_\omega$ is constant. Using the complex structure $J$, we also have a Riemannian metric $g$ related to $\omega$ by $\omega(Y, Z) = g(JY, Z)$ (see Appendix D for more details on this relation). We will often use $g$ and $\omega$ interchangeably to denote the Kähler form/metric structure on $X$.

## 1.2 Conifold Transitions

An important process in non-Kähler Calabi–Yau geometry is that of a conifold transition. This deforms one Calabi–Yau threefold $\widehat{X}$ into a family of others via a blowdown followed by a smoothing, while passing through an intermediate singular space.

### 1.2.1 Local Models

We first look at conifold transitions by studying how they act on their local models. Consider the total space of the bundle

$$\widehat{V} = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1. \tag{1.4}$$

Using the usual coordinate charts for $\mathbb{P}^1$, we have two trivializations for the space $\widehat{V}$:

$$(U, (\lambda, u, v)) \text{ and } (U', (\lambda', u', v')), \tag{1.5}$$

where $\lambda$ and $\lambda'$ are coordinates on the base $\mathbb{P}^1$ and $u$, $v$, $u'$, and $v'$ are fiber coordinates. The transition functions for these charts are

$$\lambda' = \lambda^{-1}, \qquad u' = \lambda u, \qquad v' = \lambda v. \tag{1.6}$$

This space can be regarded as a small blowup of the singular space

$$V_0 = \Big\{ z \in \mathbb{C}^4 \mid \sum_{j=1}^{4} z_j^2 = 0 \Big\}. \tag{1.7}$$

This can be seen by the holomorphic (scaled) blowdown map $\pi : \widehat{V} \to V_0$ given explicitly by

$$\pi(\lambda, u, v) = \Big( \frac{\lambda v + u}{\sqrt{2}}, \frac{-\sqrt{-1}(\lambda v - u)}{\sqrt{2}}, \frac{-\sqrt{-1}(v + \lambda u)}{\sqrt{2}}, \frac{-(v - \lambda u)}{\sqrt{2}} \Big). \tag{1.8}$$

From this expression, we can see that $\pi^{-1}(0)$ is the zero section $E \simeq \mathbb{P}^1$ of the bundle $\widehat{V}$. Away from the ordinary double point (ODP) singularity at the origin, we have

$$\begin{aligned} &\pi^{-1}(z_1, z_2, z_3, z_4) \\ &\quad = \Big( \frac{\sqrt{-1}(z_3 - \sqrt{-1} z_4)}{z_1 - \sqrt{-1} z_2}, \frac{z_1 - \sqrt{-1} z_2}{\sqrt{2}}, \frac{\sqrt{-1}(z_3 + \sqrt{-1} z_4)}{\sqrt{2}} \Big), \end{aligned} \tag{1.9}$$

which shows that the restriction $\pi_{\widehat{V} \setminus E}$ is biholomorphic.

The space $V_0$ can be check to be closed under addition and scalar multiplication and is hence a cone $[0, \infty) \times L$. The link $L$ of this cone is $S^2 \times S^3$. To see this, we write $z_j = x_j + \sqrt{-1} y_j$ to rewrite the defining condition as

$$0 = \|x\|^2 - \|y\|^2 + 2\sqrt{-1}\langle x, y\rangle \tag{1.10}$$

where $x, y \in \mathbb{R}^4$. Checking real and imaginary parts, we must have

$$\|x\|^2 = \|y\|^2 \text{ and } \langle x, y\rangle = 0. \tag{1.11}$$

Consider the set where $\|z\|^2 = 2$ and so $\|x\|^2 = \|y\|^2 = 1$. We must have that $x \in S^3 \subseteq \mathbb{R}^4$. For each choice of $x$, the above conditions imply that $y$ must be in the unit 2-sphere centered at $0 \in T_x S^3$. It follows that the set $\{z \in V_0 \mid \|z\|^2 = 2\}$ is diffeomorphic to the unit sphere bundle contained in the tangent bundle $TS^3$, which is trivial. Rescaling, we see that $L \simeq S^2 \times S^3$.

The singular space $V_0$ can be smoothed out to the spaces

$$V_t = \Big\{ z \in \mathbb{C}^4 \mid \sum_{j=1}^{4} z_j^2 = t \Big\}. \tag{1.12}$$

This is achieved by considering the maps $\Phi_t : \mathbb{C}^4 \setminus \{0\} \to \mathbb{C}^4$ defined by

$$\Phi_t(z) = z + \frac{t\overline{z}}{2\|z\|^2}, \tag{1.13}$$

where $\|\cdot\|$ is the usual norm on $\mathbb{C}^4$. Routine computations (see Appendix B) show that indeed $\Phi_t : V_0 \setminus \{0\} \to V_t$ and that the restriction

$$\Phi_t : \Big\{ z \in V_0 \mid \|z\|^2 > \frac{|t|}{2} \Big\} \to \{ z \in V_t \mid \|z\|^2 > |t| \} \tag{1.14}$$

is a diffeomorphism. We note that the condition defining $V_t$ implies that $\|z\|^2 \geq |t|$ and so the image of the restriction only excludes the set $\{z \in V_t \mid \|z\|^2 = |t|\}$.

The excluded set can be shown to be a scaled $S^3$. Indeed, using the same notation as we did when checking $L \simeq S^2 \times S^3$ and supposing that $t > 0$, we get that $\sum_j z_j^2 = t$ implies that

$$\|x\|^2 - \|y\|^2 = t, \qquad \langle x, y\rangle = 0. \tag{1.15}$$

The second condition $\sum_j \|z_j\|^2 = |t|$ then gives

$$\|x\|^2 + \|y\|^2 = t. \tag{1.16}$$

It follows that $\|x\|^2 = 2t$ and $y = 0$ as desired. We can obtain the result for other values of $t$ by rotating these appropriately. As such, we refer to this set as the vanishing sphere.

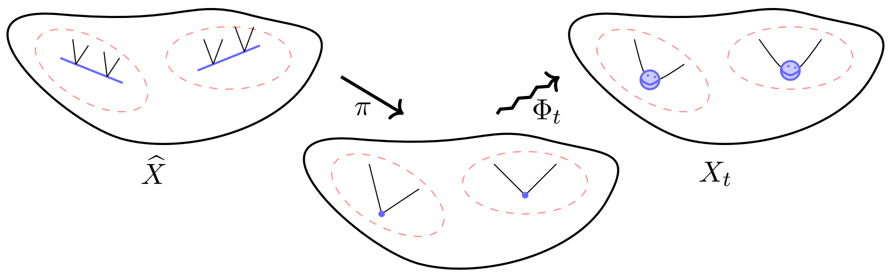


Figure 1.1: A conifold transition contracts $(-1,-1)$-curves on $\widehat{X}$ to points on $X_0$ and smooths them out to 3-spheres on $X_t$.

### 1.2.2 Global Processes and Friedman's Condition

The local blowdown and smoothing process described above is what we would like to globalize. As such, we require a special type of curve.

**Definition 1.2.1.** A $(-1,-1)$-curve $E \subseteq \widehat{X}$ is a smooth rational complex curve $E \simeq \mathbb{P}^1$ such that the normal bundle $N_{E\backslash\widehat{X}} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

Given disjoint $(-1,-1)$-curves $E_j \subseteq \widehat{X}$, we have disjoint open neighbourhoods $\widehat{U}_j \supseteq E_j$ that are biholomorphic to neighbourhoods of the zero section of the bundle $\widehat{V} = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$. This allows us to apply the local process and obtain a blowdown map $\pi : \widehat{X} \to X_0$, where $X_0$ is a singular space with isolated singular points at $s_j = \pi(E_j)$. In order to ensure that our local processes work globally, we require a homological condition given by R. Friedman.

**Theorem 1.2.2** (R. Friedman [Fri86, Fri91])**.** *Let $\widehat{X}$ be a Calabi–Yau threefold and let $E_1, \ldots, E_k \subseteq \widehat{X}$ be disjoint $(-1,-1)$-curves. Let $\pi$ be the blowdown map that contracts each $E_j$, resulting in the singular space $X_0$ with*

*ODP singularities $s_j = \pi(E_j)$. There exists a first-order deformation of $X_0$ smoothing each $s_j$ if and only if there exists a relation*

$$\sum_j \lambda_j [E_j] = 0 \text{ in } H^2(\widehat{X}, \mathbb{R}) \tag{1.17}$$

*with each $\lambda_j \neq 0$.*

It has been shown that if the $\sqrt{-1}\partial\overline{\partial}$-Lemma holds on our manifold, then these first-order deformations integrate to genuine smoothings (see [Kaw92, Ran92, Tia92]). Assuming that Friedman's Condition (Theorem 1.2.2) holds, we get a holomorphic family

$$\mu : \mathcal{X} \to \Delta \tag{1.18}$$

where $\Delta \subseteq \mathbb{C}$ denotes the complex unit disc such that $X_0 = \mu^{-1}(0)$ and the fibers $X_t = \mu^{-1}(t)$ are smooth complex manifolds for $t \neq 0$. A result of Kas–Schlessinger [KS72] shows that the family $\mathcal{X}$ is locally biholomorphic to the model space

$$\mathcal{V} = \left\{ (z,t) \in \mathbb{C}^4 \times \mathbb{C} \mid \sum_{j=1}^{4} z_j^2 = t \right\} \tag{1.19}$$

near each ODP singularity. It can be shown that the resulting manifolds $X_t$ have trivial canonical bundle and are also Calabi–Yau (see [Fri86] for an algebraic proof or [CGPY23] for a differential geometric proof).

**Remark 1.2.3.** We note here that the smoothings may only exist for $t$ with $|t|$ sufficiently small. For simplicity, we shall rescale and assume that the parameter space includes the complex unit disc $\Delta$.

Putting everything together, we can finally define a conifold transition.

**Definition 1.2.4.** Let $\widehat{X}$ be a Calabi–Yau threefold. A conifold transition starting from $\widehat{X}$, denoted $\widehat{X} \to X_0 \rightsquigarrow X_t$, consists of a holomorphic map $\pi : \widehat{X} \to X_0$ and a family $\mu : \mathcal{X} \to \Delta$ with $X_0 = \mu^{-1}(0)$ such that

i) the map $\pi : \widehat{X} \to X_0$ contracts a collection of disjoint $(-1,-1)$-curves $E_1, \ldots, E_k \subseteq \widehat{X}$ to isolated ODP singularities $s_1, \ldots, s_k \in X_0$ and $\pi$ is a biholomorphism between $\widehat{X} \setminus (E_1 \cup \ldots \cup E_k)$ and $X_0 \setminus \{s_1, \ldots, s_k\}$; and

ii) the total space $\mathcal{X}$ is a smooth complex fourfold with a proper flat map $\mu : \mathcal{X} \to \Delta$, where $X_0 = \mu^{-1}(0)$ and each $X_t = \mu^{-1}(t)$ is a smooth complex threefold for $t \neq 0$.

### 1.2.3 Topological Changes

From the local geometry, we see that conifold transitions contract 2-cycles from the small resolution $\widehat{X}$ and generate 3-cycles on the smoothings. These topological differences correspond to changes in Betti numbers $b_k$, in particular if a conifold transition $\widehat{X} \to X_0 \rightsquigarrow X_t$ contracts $N$ curves spanning $\ell$ dimensions in homology, then

$$b_2(X_t) = b_2(\widehat{X}) - \ell, \qquad b_3(X_t) = b_3(\widehat{X}) + 2(N - \ell). \tag{1.20}$$

A consequence of the above is that conifold transitions may not preserve the Kähler condition.

**Example 1.2.5.** Consider the projective quintic

$$\widehat{X} = \left\{ [z_0 : \ldots : z_4] \in \mathbb{P}^4 \mid \sum_{j=0}^{4} z_j^5 = 0 \right\}. \tag{1.21}$$

Since $\widehat{X}$ is projective, it admits a Kähler metric and it can be shown that $b_2(\widehat{X}) = 1$. If we contract two linearly dependent $(-1,-1)$-curves $E_1, E_2$ on $\widehat{X}$, we get that $N = 2$ and $\ell = 1$. From this, we see that $b_2(X_t) = 0$ and as such, the smoothings $X_t$ cannot admit Kähler structures.

The above example presents an issue that the Kähler condition is incompatible with conifold transitions. An idea of Friedman [Fri91] and Reid [Rei87] to get around this is to include any spaces that can be reached from a Kähler Calabi–Yau threefold in our central class of objects, which leaves us with the following central open questions:

**Question 1.2.6.** What spaces can be reached from a Kähler Calabi–Yau threefold via conifold transitions? What properties do these space have? What geometry should we endow these spaces with?

## 1.3 The Hull–Strominger System

A major conjecture to these open questions comes in the form of Reid's Fantasy, which conjectures that all Calabi–Yau threefolds can be linked by a sequence of conifold transitions [Rei87]. This has since been verified for large classes of examples (see *e.g.,* [ACJM96, CGH90]). Alongside the fact that conifold transitions do not preserve the Kähler condition, we see that a (Ricci-flat) Kähler metric is not the "correct" geometry for these spaces. A conjecture of Yau says that the model geometry should come from a pair of compatible metrics with the relevant compatibility coming from the Hull–Strominger system, a system of PDEs originating from heteroric string theory which describe conditions for supersymmetry.

**Definition 1.3.1.** Let $X$ be a compact complex threefold with nowhere-vanishing holomorphic $(3,0)$-form $\Upsilon$ and let $E \to X$ be a holomorphic vector bundle. Fix a constant $\alpha' > 0$. A solution to the Hull–Strominger system with slope parameter $\alpha'$ consists of a pair of Hermitian metrics $\omega$ on $X$ and $H$ on $E$ such that

$$F^{2,0} = F^{0,2} = 0, \qquad \omega \wedge F^{1,1} = 0, \tag{1.22}$$

$$\sqrt{-1}\partial\overline{\partial}\omega - \alpha' \cdot \Big(\mathrm{tr}(\mathrm{Rm} \wedge \mathrm{Rm}) - \mathrm{tr}(F \wedge F)\Big) = 0, \tag{1.23}$$

$$d^{\dagger}\omega = \sqrt{-1}(\overline{\partial} - \partial) \log \|\Upsilon\|_{\omega}, \tag{1.24}$$

where Rm and $F$ are the Chern curvatures of $\omega$ and $H$ respectively.

Proposed by Hull [Hul86] and Strominger [Str86], these equations generalize the compactification of the 10D heterotic string in [CHSW85]. The first of these equations is a Hermitian Yang–Mills (HYM) relation between the metrics $\omega$ and $H$. The second equation is called the heterotic Bianchi identity and comes from the Green–Schwarz anomaly cancellation [GS84]. The third equation, which at first glance seems to be a relation between the torsion of the metric $\omega$, was shown by Li–Yau [LY05] to be equivalent to the conformally balanced condition

$$d\big(\|\Upsilon\|_{\omega}\omega^2\big) = 0. \tag{1.25}$$

A proof of their result can be found in Appendix A.

**Remark 1.3.2.** Some authors may use other connections instead of the Chern connection in the definition of the Hull–Strominger system and the Anomaly flow in §4. Both of these can also be generalized to other dimensions (see *e.g.,* [PPZ19a]) and to manifolds with other special structures (see *e.g.,* [CGFT22, dlOLS18, dSGFLSE24, FIUV15, II05]). We will, however, restrict our attention to the complex dimension 3 setting and later, the $G_2$ setting in §5.2.

Using this, we have the aforementioned Reid's Fantasy with modifications by Yau.

**Conjecture 1.3.3** (Reid [Rei87], Yau)**.** All Calabi–Yau threefolds can be linked by a sequence of conifold transitions. Further, each Calabi–Yau threefold admits a unique solution to the Hull–Strominger system (1.22) - (1.25) in a suitable cohomology class.

Since our main objects of study includes Kähler Calabi–Yau threefolds, the proposed geometry should generalize the notion of a (Ricci-flat) Kähler metric. We see this in the example below.

**Example 1.3.4.** Let $X$ be a Käher Calabi–Yau threefold with Kähler metric $\omega$ and holomorphic $(3,0)$-form $\Upsilon$. By Yau's Theorem, there exists a unique Ricci-flat Kähler metric $\omega_{\mathrm{CY}} \in [\omega]$. Let $E = T^{1,0}X$ be the holomorphic tangent bundle of $X$ and set both connections $\omega$ and $H$ equal to $\omega_{\mathrm{CY}}$. Since $\omega_{\mathrm{CY}}$ is Ricci-flat, we get that $H$ is Hermitian Yang–Mills with respect to $\omega$ and also that the norm $\|\Upsilon\|_\omega = \|\Upsilon\|_{\omega_{\mathrm{CY}}}$ is constant. The Kähler condition tells us that $\omega_{\mathrm{CY}}$ and hence $\omega_{\mathrm{CY}}^2$ is closed. It follows that the pair $(\omega_{\mathrm{CY}}, \omega_{\mathrm{CY}})$ of Hermitian metrics solve (1.22) - (1.25) for any choice of slope parameter $\alpha'$, as expected.

# Chapter 2

# Geometrizing Conifold Transitions

As conjectured by Reid's Fantasy (Conjecture 1.3.3) we expect that there exist solutions to the Hull–Strominger system (1.22) - (1.25) on both sides of a conifold transition $\widehat{X} \to X_0 \rightsquigarrow X_t$. In this chapter, we discuss the problem of geometrizing conifold transitions and the metric constructions of Fu–Li–Yau [FLY12] and Collins–Picard–Yau [CPY24].

## 2.1 Metrics on the Small Resolutions, Cones, and Conifolds

### 2.1.1 Candelas–de la Ossa Metrics on the Local Model

Recall our model space

$$\widehat{V} = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1 \tag{2.1}$$

with trivializations

$$(U, (\lambda, u, v)) \text{ and } (U', (\lambda', u', v')). \tag{2.2}$$

We have a well-defined radius function $r : \widehat{V} \to [0, \infty)$ given by

$$r(\lambda, u, v) = (1 + |\lambda|^2)^{\frac{1}{3}} \cdot (|u|^2 + |v|^2)^{\frac{1}{3}}. \tag{2.3}$$

Without the exponent of $\frac{1}{3}$, this function measures the distance from a point to the zero section $E \simeq \mathbb{P}^1$ along the fibers using the Fubini–Study metric $\widehat{\omega}_{\text{FS}}$. The exponent is introduced so that the radius function coincides with the radius of the Calabi–Yau cone metric on the blowdown.

We can equip the space $\widehat{V}$ with a family of scaling maps $S_R : \widehat{V} \to \widehat{V}$ for $R > 0$ defined by

$$S_R(\lambda, u, v) = (\lambda, R^{\frac{3}{2}} \cdot u, R^{\frac{3}{2}} \cdot v). \tag{2.4}$$

These maps are compatible with the radius function $r$ as

$$r \circ S_R = R \cdot r. \tag{2.5}$$

In [CdlO90], Candelas–de la Ossa look for a family of Ricci-flat Kähler metrics $\widehat{\omega}_{\mathrm{co},a}$ on $\widehat{V}$ of the form

$$\widehat{\omega}_{\mathrm{co},a} = \sqrt{-1}\partial\overline{\partial} f_a(r^3) + 4a^2\widehat{\omega}_{\mathrm{FS}}, \tag{2.6}$$

where $f_a(x) = f_a(r^3)$ is a smooth function. By imposing the Ricci-flat condition, they obtain a first-order ODE for $f_a$:

$$x(f_a'(x))^3 + 6a^2(f_a'(x))^2 = 1. \tag{2.7}$$

We note that if $f_1(x)$ is a solution for $a = 1$, then $f_a(x) = a^2 \cdot f_1(\frac{x}{a^3})$ is a solution for arbitrary $a > 0$.

The solution for $a > 0$ admits an expansion [CPY24] for $x \gg 1$ given in terms of $r = x^{\frac{1}{3}}$ by

$$f_a(r^3) = c_0 \cdot r^2 + c_1 \cdot a^2 \log(a^{-3}r) + c_2 \cdot a^4 r^{-2} + c_3 \cdot a^6 r^{-4} + \dots \tag{2.8}$$

for constants $c_0, c_1, c_2, c_3, \dots$. Thus after rescaling $\widehat{\omega}_{\mathrm{co},1}$ such that $c_0 = \frac{1}{2}$, we have the following expansion for large radius $r \gg 1$:

$$\begin{aligned}\widehat{\omega}_{\mathrm{co},a} - \frac{1}{2}\sqrt{-1}\partial\overline{\partial} r^2 &= c_{-1} \cdot a^2\widehat{\omega}_{\mathrm{FS}} + c_1 \cdot a^2\sqrt{-1}\partial\overline{\partial}\log r \\ &\quad + c_2 \cdot a^4\sqrt{-1}\partial\overline{\partial} r^{-2} + c_3 \cdot a^6\sqrt{-1}\partial\overline{\partial} r^{-4} + \dots . \end{aligned} \tag{2.9}$$

When $a = 0$, we instead have $f_0 = \frac{1}{2}r^2$ and get the metric

$$\widehat{\omega}_{\mathrm{co},0} = \frac{1}{2}\sqrt{-1}\partial\overline{\partial} r^2, \tag{2.10}$$

which is singular on the zero section $E$.

Using the (scaled) blowdown map $\pi : \widehat{V} \to V_0$ given by

$$\begin{aligned}&\pi(\lambda, u, v) \\ &\quad = \Big(\frac{\lambda v + u}{\sqrt{2}}, \frac{-\sqrt{-1}(\lambda v - u)}{\sqrt{2}}, \frac{-\sqrt{-1}(v + \lambda u)}{\sqrt{2}}, \frac{-(v - \lambda u)}{\sqrt{2}}\Big),\end{aligned} \tag{2.11}$$

we get that the radius function $r$ on $\widehat{V}$ becomes $\|z\|^{\frac{2}{3}}$ on $V_0$ in the sense that

$$r(\lambda, u, v) = \|\pi(\lambda, u, v)\|^{\frac{2}{3}}. \tag{2.12}$$

As such, we will define another radius function $r : V_0 \to [0, \infty)$ by

$$r(z) = \|z\|^{\frac{2}{3}}. \tag{2.13}$$

The singular space $V_0$ admits a Calabi–Yau cone metric

$$\omega_{\mathrm{co},0} = \frac{1}{2}\sqrt{-1}\partial\overline{\partial}r^2 \tag{2.14}$$

which is well known [CdlO90] to be Ricci-flat Kähler and is a cone metric over the link $L = \{z \in V_0 \mid r(z) = 1\} \simeq S^2 \times S^3$. As such, we write

$$g_{\mathrm{co},0} = dr \otimes dr + r^2 \cdot g_L \tag{2.15}$$

where $g_L$ is the pullback of a metric on $L$. The metrics $\widehat{g}_{\mathrm{co},0}$ and $g_{\mathrm{co},0}$ coincide away from the zero section and singularity after identification by pullback.

Returning to the space $\widehat{V}$, we obtained a 1-parameter family of metrics $\widehat{g}_{\mathrm{co},a}$ for $0 \leq a \leq 1$ which we will refer to as the Candelas–de la Ossa metrics on the small resolution. This family of metrics satisfies two important properties:

(CO SR I) *Normalization:* For $0 < a \leq 1$, we have

$$\widehat{g}_{\mathrm{co},a} = a^2 \cdot S^*_{a^{-1}}(\widehat{g}_{\mathrm{co},1}). \tag{2.16}$$

(CO SR II) *Asymptotically Conical Decay:* There exists $C > 0$ independent of $a$ such that for all $0 < a \leq 1$,

$$\|(\pi^{-1})^*(\widehat{g}_{\mathrm{co},a}) - g_{\mathrm{co},0}\|_{g_{\mathrm{co},0}} \leq C \cdot a^2 r^{-2}. \tag{2.17}$$

The asymptotic decay can be derived from (2.9) for $a = 1$. Pulling back the estimate when $a = 1$ back by $S^*_a$ gives the estimate for general non-zero $a$. The estimate implies that the Candelas–de la Ossa metrics $\widehat{g}_{\mathrm{co},a}$ converge uniformly to the cone metric $g_{\mathrm{co},0}$ on compact sets away from the zero section $E$.

In the sequel, we will work with neighbourhoods of the zero section in the local model $\widehat{V}$. As such, for $R > 0$, we define "tubular" neighbourhoods of radius $R$ by

$$\widehat{T}(R) := \{r \leq R\} \subseteq \widehat{V}. \tag{2.18}$$

Likewise, on the cone $V_0$, we define the "disc" of radius $R$ about the origin by

$$D_0(R) := \{r \leq R\} \subseteq V_0. \tag{2.19}$$

### 2.1.2 Balanced Metrics on the Small Resolution

Under the assumption that the original Calabi–Yau threefold $\widehat{X}$ admits a Kähler metric $\omega$, Fu–Li–Yau [FLY12] constructed a family of balanced metrics $\widehat{\omega}_{\mathrm{FLY},a}$ for $0 \le a \le 1$ on $\widehat{X}$ based on the Candelas–de la Ossa local metrics $\widehat{\omega}_{\mathrm{co},a}$. This was achieved using a gluing method that interpolates between the local metrics around contracted $(-1,-1)$-curves and the ambient Ricci-flat Kähler metric $\widehat{\omega}_{\mathrm{CY}} \in [\omega]$ which exists by Yau's Theorem (see also [CPY24, Chu12] for further details on the metrics $\widehat{\omega}_{\mathrm{FLY},a}$). Their construction heavily exploits the known nature of the model neighbourhoods and produces metrics such that

$$d\widehat{\omega}^2_{\mathrm{FLY},a} = 0, \qquad [\widehat{\omega}^2_{\mathrm{FLY},a}] = [\widehat{\omega}^2_{\mathrm{CY}}] \in H^4(\widehat{X}, \mathbb{R}). \tag{2.20}$$

For our purposes, we mainly make use of the following two properties:

(FLY SR I) *Local Model:* There exists $\delta > 0$ and $R > 1$ such that for all $0 \le a \le 1$, we have

$$\widehat{\omega}_{\mathrm{FLY},a}|_{\{r<\delta\}} = R \cdot \widehat{\omega}_{\mathrm{co},a}. \tag{2.21}$$

Here the function $r : \widehat{X} \to [0,\infty)$ extends the local functions $r$ defined on a neighborhood of the curves $E_j \subset \widehat{V}$ to the whole compact manifold $\widehat{X}$ such that the set $\{r < \delta\}$ consists of small disjoint open neighborhoods containing the $(-1,-1)$-curves $E_1, \ldots, E_k$.

(FLY SR II) *Uniform Convergence:* For any compact set $K \subset \widehat{X} \setminus (E_1 \cup \cdots \cup E_k)$, the sequence $\widehat{\omega}_{\mathrm{FLY},a}$ converges uniformly to $\widehat{\omega}_{\mathrm{FLY},0}$ as $a \to 0$ on $K$.

For each $E_j \simeq \mathbb{P}^1$, these metrics satisfy

$$\int_{\mathbb{P}^1} \widehat{\omega}_{\mathrm{FLY},a} \to 0 \text{ as } a \to 0. \tag{2.22}$$

The limiting metric $\widehat{\omega}_{\mathrm{FLY},0}$ is singular on the curves $E_1, \ldots, E_k$ and only defines a genuine metric on $\widehat{X} \setminus (E_1 \cup \ldots \cup E_k)$.

Let $\pi : \widehat{X} \to X_0$ be the blowdown map contracting the curves $E_1, \ldots, E_k$ and $s_j = \pi(E_j)$ be the singular points of $X_0$. We write $(X_0)_{\mathrm{reg}} = X_0 \setminus \{s_1, \ldots, s_k\}$ to denote the regular part of $X_0$. Since $\widehat{X} \setminus (E_1 \cup \ldots \cup E_k) \simeq (X_0)_{\mathrm{reg}}$, the limiting metric $\widehat{\omega}_{\mathrm{FLY},0}$ defines a Riemannian structure $((X_0)_{\mathrm{reg}}, \omega_{\mathrm{FLY},0})$ with conical singularities.

### 2.1.3 Hermitian Yang–Mills Metrics on the Small Resolution

Using the balanced metrics $\widehat{\omega}_{\mathrm{FLY},a}$ of Fu–Li–Yau and the additional assumption that $\widehat{X}$ is also simply connected, Collins–Picard–Yau [CPY24] were able to construct another family of metrics that satisfy the Hermitian Yang–Mills condition with respect to $\widehat{\omega}_{\mathrm{FLY},a}$. We briefly discuss their construction here. Recall that the initial manifold $\widehat{X}$ is Kähler Calabi–Yau with some Ricci-flat Kähler metric $\widehat{\omega}_{\mathrm{CY}}$.

As noted in [Yau93], by the de Rham Decomposition Theorem we have that if the tangent bundle $T^{1,0}\widehat{X}$ splits holomorphically, then $\widehat{X}$ itself also splits holomorphically as a product. By dimensional considerations, at least one factor in this decomposition must be complex 1-dimensional and compactness requires that this be a torus. The simply connected condition rules this possibility out.

Since $\widehat{\omega}_{\mathrm{CY}}$ is Ricci-flat Kähler on $\widehat{X}$, we have that $T^{1,0}\widehat{X}$ is polystable with repect to $[\widehat{\omega}_{\mathrm{CY}}]$. As such, we must have that $(\widehat{X}, \widehat{\omega}_{\mathrm{CY}})$ satisfies the stability condition

$$\frac{1}{\mathrm{rank} F} \int_{\widehat{X}} c_1(F) \wedge \widehat{\omega}_{\mathrm{CY}}^2 < 0 \tag{2.23}$$

for all torsion-free coherent proper subsheaves $F \subseteq T^{1,0}\widehat{X}$. The Fu–Li–Yau metrics $\widehat{\omega}_{\mathrm{FLY},a}$ share the same square cohomology class with $\widehat{\omega}_{\mathrm{CY}}$ (2.20). Hence

$$\frac{1}{\mathrm{rank} F} \int_{\widehat{X}} c_1(F) \wedge \widehat{\omega}_{\mathrm{FLY},a}^2 < 0 \tag{2.24}$$

and so the bundle $T^{1,0}\widehat{X}$ is stable with respect to each of the Fu–Li–Yau metrics $\widehat{\omega}_{\mathrm{FLY},a}$. The Li–Yau [LY87] generalization of the Donaldson–Uhlenbeck–Yau Theorem [Don85, UY86] for Gauduchon metrics yields a family of Hermitian metrics $\widehat{H}_a$ satisfying

$$F_{\widehat{H}_a} \wedge \widehat{\omega}_{\mathrm{FLY},a}^2 = 0, \qquad \int_{\widehat{X}} \log\left(\frac{\det \widehat{H}_a}{\det \widehat{g}_{\mathrm{FLY},a}}\right) d\mathrm{vol}_{\widehat{g}_{\mathrm{FLY},a}} = 0. \tag{2.25}$$

The sequence of metrics $\widehat{H}_a$ satisfies the following estimates:

**Proposition 2.1.1** (Collins–Picard–Yau [CPY24])**.** *There exist constants $C, C_p > 0$ for each $p \geq 0$ such that the metrics $\widehat{H}_a$ satisfy*

$$C^{-1} \cdot \widehat{g}_{\mathrm{FLY},a} \leq \widehat{H}_a \leq C \cdot \widehat{g}_{\mathrm{FLY},a}, \tag{2.26}$$

$$\|\nabla^p \widehat{H}_a\|_{\widehat{g}_{\mathrm{FLY},a}} \leq C_p \cdot r^{-p}. \tag{2.27}$$

The metric $H_0$ on $(X_0)_{\mathrm{reg}}$ can be constructed as the limit of the $\widehat{H}_a$. This was done in [CPY24] by taking a subsequence of $\{\widehat{H}_a\}$, however the estimates in Proposition 2.1.1 actually imply that the full sequence converges on compact sets (see Appendix A of [FPS24] for the full argument). As such, there exists a Hermitian Yang–Mills metric $H_0$ over the singular space $X_0$ such that

$$F_{H_0} \wedge \omega_{\mathrm{FLY},0}^2 = 0, \qquad C^{-1} \cdot g_{\mathrm{FLY},0} \leq H_0 \leq C \cdot g_{\mathrm{FLY},0}. \tag{2.28}$$

Further, we also have that the sequence $\widehat{H}_a$ converges uniformly to $H_0$ on any compact set $K \subseteq \widehat{X} \setminus (E_1 \cup \ldots \cup E_k)$ as $a \to 0$.

## 2.2 Metrics on the Smoothings

### 2.2.1 Candelas–de la Ossa Metrics on the Local Model

Candelas–de la Ossa [CdlO90] also constructed Ricci-flat Kähler metrics $\omega_{\mathrm{co},t}$ on the local models

$$V_t = \left\{ z \in \mathbb{C}^4 \mid \sum_{j=1}^{4} z_j^2 = t \right\} \tag{2.29}$$

for non-zero $t$. Here, we have the radius function $r : V_t \to [0, \infty)$

$$r(z) = \|z\|^{\frac{2}{3}} \tag{2.30}$$

matching the definition of those from the previous section. The Ansatz that Candelas–de la Ossa used for these spaces were likewise of the form

$$\omega_{\mathrm{co},t} = \sqrt{-1}\partial\overline{\partial} f_t(r^3), \tag{2.31}$$

where $f_t(x) = f_t(r^3)$ is a smooth function. By once again imposing the Ricci-flat condition, they arrive at another ODE for $f_t$:

$$x(f_t'(x))^3 + (f_t'(x))^2 f_t''(x - |t|^2) = \frac{1}{6}. \tag{2.32}$$

For $t = 0$, the solution is proportional to

$$f_0(r^3) = r^2 \tag{2.33}$$

and for non-zero $t$, the solution is proportional to

$$f_t(r^3) = \Big(\frac{|t|^2}{2}\Big)^{\frac{1}{3}} \int_0^{\cosh^{-1}(\frac{r^3}{|t|})} \big(\sinh(2\tau) - 2\tau\big)^{\frac{1}{3}} d\tau. \tag{2.34}$$

Like the metrics $\widehat{\omega}_{\mathrm{co},a}$ on the small resolution, the metrics $\omega_{\mathrm{co},t}$ are also asymptotic to the cone geometry $(V_0, \omega_{\mathrm{co},0})$.

Even though the spaces $V_t$ are distinct, we still have scaling maps by considering the larger space $\mathbb{C}^4$. Given $R \neq 0$, we define $S_R : \mathbb{C}^4 \to \mathbb{C}^4$ by

$$S_R(z) = R^{\frac{3}{2}} \cdot z. \tag{2.35}$$

The map $S_R$ sends $V_\rho$ to $V_{R^3 \cdot \rho}$ and satisfies

$$r \circ S_R = R \cdot r \text{ and } S_R^*(\omega_{\mathrm{co},0}) = R^2 \cdot \omega_{\mathrm{co},0}. \tag{2.36}$$

In order to compare $\omega_{\mathrm{co},t}$ to the cone metric $\omega_{\mathrm{co},0}$, we pullback by the map $\Phi_t : \mathbb{C}^4 \setminus \{0\} \to \mathbb{C}^4$ from §1.2.1, which was defined by

$$\Phi_t(z) = z + \frac{t\overline{z}}{2\|z\|^2}. \tag{2.37}$$

One can show that the scaling maps and the maps $\Phi_t$ are related by

$$\Phi_t = S_{t^{\frac{1}{3}}} \circ \Phi_1 \circ S_{t^{-\frac{1}{3}}}. \tag{2.38}$$

The radius functions interact with $\Phi_t$ by

$$r(\Phi_t(z)) = \left( \big(r(z)\big)^3 + \frac{|t|^2}{4\big(r(z)\big)^3} \right)^{\frac{1}{3}}. \tag{2.39}$$

As such, for convenience, we define

$$\beta_{t,\rho} = \Big(\rho^3 + \frac{|t|^2}{4\rho^3}\Big)^{\frac{1}{3}}. \tag{2.40}$$

Using the maps $S_R$ and $\Phi_t$, we have the analogous properties of the Candelas–de la Ossa metrics $g_{\mathrm{co},t}$ on the smoothings:

(CO SM I) *Normalization:* For $t \neq 0$, we have

$$g_{\mathrm{co},t} = |t|^{\frac{2}{3}} \cdot S^*_{t^{-\frac{1}{3}}}(g_{\mathrm{co},1}). \tag{2.41}$$

(CO SM II) *Asymptotically Conical Decay:* There exists a constant $C > 0$ independent of $t$ such that for all $t \neq 0$,

$$\|(\Phi_t)^*(g_{\mathrm{co},t}) - g_{\mathrm{co},0}\|_{g_{\mathrm{co},0}} \leq C \cdot |t| r^{-3}. \tag{2.42}$$

A consequence of this is that the metrics $g_{\mathrm{co},t}$ approach $g_{\mathrm{co},0}$ on compact sets away from the vanishing spheres $\{z \in V_t \mid \|z\|^2 = |t|\}$ as $t \to 0$.

The proof of the asymptotically conical decay estimate can be found in [CH13], where the estimate is given on $(V_1, g_{\mathrm{co},1})$:

$$\|(\Phi_1)^*(g_{\mathrm{co},1}) - g_{\mathrm{co},0}\|_{g_{\mathrm{co},0}} \leq C \cdot r^{-3}. \tag{2.43}$$

The generic estimate for $(V_t, g_{\mathrm{co},t})$ follows by pulling back by $S_{t^{-\frac{1}{3}}}$.

As we did for the small resolution $\widehat{V}$ and the cone $V_0$, we also define the "disc" of radius $R > 0$ about the origin in $V_t$ by

$$D_t(R) := \{r \leq R\} \subseteq V_t. \tag{2.44}$$

### 2.2.2 Balanced Metrics on the Smoothings

We return to the global setting, where we have a holomorphic family $\mu : \mathcal{X} \to \Delta$ with smooth fibers $X_t = \mu^{-1}(t)$ for $t \neq 0$ and central fiber $X_0 = \mu^{-1}(0)$ with singularities $\{s_1, \ldots, s_k\}$ which are locally of the from $0 \in V_0$. Even though Example 1.2.5 shows that the compact complex manifolds $X_t$ may not admit Kähler metrics, Fu–Li–Yau [FLY12] prove that they admit balanced metrics.

As we did in the case of the small resolutions, we first have to extend the local maps $r$ and $\Phi_t$ to global objects. To do this, we note that there are disjoint open sets $\mathcal{U}_j \subseteq \mathcal{X}$ containing each singularity $s_j$ such that $\mathcal{U}_j$ is identified with

$$0 \in \mathcal{U} \subseteq \Big\{(z,t) \in \mathbb{C}^4 \times \mathbb{C} \mid \sum_{j=1}^{4} z_j^2 = t\Big\}. \tag{2.45}$$

We can then extend the local functions $r(z) = \|z\|^{\frac{2}{3}}$ on $\mathbb{C}^4$ to a global function $r : \mathcal{X} \to [0, \infty)$ with $r^{-1}(0) = \{s_1, \ldots, s_k\}$.

Next, we can extend the local maps $\Phi_t$ to global diffeomorphisms

$$\Phi_t : X_0 \cap \left\{r(z) > \left(\frac{|t|}{2}\right)^{\frac{1}{3}}\right\} \to X_t \cap \{r(z) > |t|^{\frac{1}{3}}\} \tag{2.46}$$

such that $\Phi_t$ is the model smoothing on the local sets $\mathcal{U}_j$. This can be done by taking a horizontal lift $\xi$ of the vector field $\frac{\partial}{\partial t}$ which agrees with the vector field generating the model smoothing. By flowing by the lift $\xi$, we obtain $\Phi_t$.

The Fu–Li–Yau construction [FLY12] gives a sequence $\omega_{\mathrm{FLY},t}$ of Hermitian metrics on $X_t$ solving

$$d\omega^2_{\mathrm{FLY},t} = 0. \tag{2.47}$$

These are obtained first by a pullback and gluing construction followed by a perturbation to ensure the balanced condition holds. The first step yields auxiliary metrics $g_{\mathrm{aux},t}$ while the second produces the final metrics $g_{\mathrm{FLY},t}$.

We briefly outline this process below.

- Step 1: The expression for $\omega_{\mathrm{aux},t}$ from [FLY12] is

$$\omega^2_{\mathrm{aux},t} = \mathrm{pr}^{2,2}_t \Bigg[ (\Phi_t^{-1})^* \Big( \omega^2_{\mathrm{FLY},0} - \sqrt{-1}\partial\overline{\partial}\big(\rho_0 \cdot f_0(r^3) \cdot \sqrt{-1}\partial\overline{\partial} f_0(r^3)\big)\Big) + \sqrt{-1}\partial\overline{\partial}\Big(\rho_t \cdot f_t(r^3) \cdot \sqrt{-1}\partial\overline{\partial} f_t(r^3)\Big)\Bigg], \tag{2.48}$$

  where the operator $\mathrm{pr}^{2,2}_t$ denotes the projection onto the $(2,2)$-component with respect to the complex structure $J_t$ on $X_t$, the functions $\rho_0$ and $\rho_t$ are smooth cutoff functions, and the functions $f_0$ and $f_t$ are as in (2.33) and (2.34).

  These functions come from the local model $\omega_{co,t} = \sqrt{-1}\partial\overline{\partial} f_t(r^3)$, so that in a neighbourhood $\{r < \delta\}$, where the cutoff functions are identically 1, we have $\omega^2_{\mathrm{aux},t} = \omega^2_{\mathrm{co},t}$. The metric $\omega_{\mathrm{aux},t}$ is obtained by taking a square root of (2.48) (see *e.g.,* [Mic82]).

- Step 2: The perturbations correct the auxiliary metrics $\omega_{\mathrm{aux},t}$ by setting

$$\omega^2_{\mathrm{FLY},t} = \omega^2_{\mathrm{aux},t} + \partial\overline{\partial}^\dagger\partial^\dagger\gamma_t - \overline{\partial}\partial^\dagger\overline{\partial}^\dagger\overline{\gamma}_t, \tag{2.49}$$

  where $\gamma_t \in \Lambda^{2,3}(X_t)$ solves

$$E_t(\gamma_t) = \overline{\partial}\omega^2_{\mathrm{aux},t}, \qquad d\gamma_t = 0. \tag{2.50}$$

  Here $E_t$ denotes the Kodaira–Spencer operator [KS60], which is a 4th-order elliptic operator which acts on $(2,3)$-forms by

$$E_t = \partial\overline{\partial}\overline{\partial}^\dagger\partial^\dagger + \partial^\dagger\overline{\partial}\overline{\partial}^\dagger\partial + \partial^\dagger\partial. \tag{2.51}$$

The adjoints in the definition are taken with respect to the auxiliary metrics $\omega_{\text{aux},t}$.

The construction is made such that $d\omega^2_{\text{FLY},t} = 0$, and the main part of the argument in [FLY12] is to prove that $\omega_{\text{FLY},t} > 0$ when $|t|$ is sufficiently small.

The two properties of these metrics that we shall need are the following:

(FLY SM I) *Local Model:* Near each singular point $s_j \in \mathcal{X}$, there exist constants $C, R_0, c_j > 0$ and such that if $|t|$ is sufficiently small, then

$$\sup_{\{r \leq R_0\}} \|g_{\text{FLY},t} - c_j \cdot g_{\text{co},t}\|_{g_{\text{co},t}} \leq C \cdot |t|^{\frac{2}{3}}. \tag{2.52}$$

(FLY SM II) *Uniform Convergence:* For any compact set $K \subseteq (X_0)_{\text{reg}}$, the sequence $\Phi_t^*(g_{\text{FLY},t})$ converges uniformly to $g_{\text{FLY},0}$ as $t \to 0$ on $K$.

These properties can be extracted from the estimates in [FLY12], and we refer to [CPY24] for further discussion.

### 2.2.3 Hermitian Yang–Mills Metrics on the Smoothings

In order to get approximate Hermitian Yang–Mills solutions on the smoothings, Collins–Picard–Yau [CPY24] glued the pullback of the metric $H_0$ to the Candelas–de la Ossa metrics. These approximate metrics were perturbed to obtain true solutions $H_t$ to the Hermitian Yang–Mills equations. The resulting metrics $H_t$ on $X_t$ solve

$$F_{H_t} \wedge \omega^2_{\text{FLY},t} = 0, \qquad \int_{X_t} \log\left(\frac{\det H_t}{\det g_{\text{FLY},t}}\right) d\text{vol}_{g_{\text{FLY},t}} = 0. \tag{2.53}$$

Their construction is such that

$$H_t = e^u \cdot H_{\text{aux},t}, \qquad \|u\|_{H_{\text{aux},t}} + r\|\nabla u\|_{H_{\text{aux},t}} \leq C \cdot |t|^{\frac{\beta}{3}} \tag{2.54}$$

for some $\beta \in (0,1)$, and

$$H_{\text{aux},t} = \chi \cdot g_{\text{co},t} + (1-\chi) \cdot \left[(\Phi_t^{-1})^* H_0\right]^{1,1}. \tag{2.55}$$

Here, the superscript $(1,1)$ denotes the $J_t$-invariant component, $\chi = \zeta(|t|^{-\alpha} r^3)$ for some $\alpha \in (0,1)$ and $\zeta : [0,\infty) \to [0,1]$ is a cutoff function with $\zeta \equiv 1$ on $[0,1]$ and $\zeta \equiv 0$ on $[2,\infty)$.

The metrics $H_t$ are uniformly equivalent to $g_{\text{FLY},t}$, so that

$$C^{-1} \cdot g_{\text{FLY},t} \leq H_t \leq C \cdot g_{\text{FLY},t}. \tag{2.56}$$

Furthermore, for any compact set $K \subseteq (X_0)_{\text{reg}}$, the sequence $\Phi_t^*(H_t)$ converges uniformly to $H_0$ on $K$ as $t \to 0$.

## 2.3 Cones and Conifolds as Complete Metric Spaces

Much of the previous sections was devoted to constructing families of metrics for the Riemannian manifolds $(\widehat{V}, \widehat{g}_{\text{co},a})$, $(V_t, g_{\text{co},t})$, $(\widehat{X}, \widehat{g}_{\text{FLY},a})$, and $(X_t, \widehat{g}_{\text{FLY},t})$. In these cases, the Riemannian metric naturally induces the structure of a distance function, and so we can consider each of these as complete metric spaces. To do the same for the singular spaces $V_0$ and $X_0$, we need to take some care in this process.

Given a cone $V_0$ with link $L$ equipped with a cone metric

$$g_0 = dr \otimes dr + r^2 \cdot g_L \tag{2.57}$$

on $(V_0)_{\text{reg}}$, we can define a distance function $d_0$ on all of $V_0$ by extending $g_0$ to the singularity $s$ by taking

$$g_0|_s = 0. \tag{2.58}$$

Even though this extension is neither continuous nor positive-definite, since the singularity is a point, this will have no effect on the lengths of curves. Hence we can define $d_0$ in the usual way, via the infimum of lengths of piecewise smooth curves, without issue.

We can extend this idea to a conifold $X_0$ with smooth metric $g_0$ on $(X_0)_{\text{reg}}$ that satisfies

$$g_0 \leq C_j \cdot (dr \otimes dr + r^2 \cdot g_L) \tag{2.59}$$

in a neighbourhood of each isolated singularity $s_j$. First, extend $g_0$ to all of $X_0$ by setting

$$g_0|_{s_j} = 0 \tag{2.60}$$

at each singular point $s_j$. The distance function $d_0$ on $X_0$ can then be defined via integration over curves, and the distance between any two points in $X_0$ is finite. As a result, we can endow $X_0$ with the structure of a compact length space, whose admissible curves are exactly the piecewise differentiable curves on $X_0$. A similar construction to this is done in [SW13a, SW14].

## 2.4 Notation and Conventions

Before we continue with our study of conifold transitions, we establish a general notational guideline due to the sheer number of metrics and distance functions involved:

- When working with quantities related to metrics on small resolutions ($\widehat{X}$ and $\widehat{V}$), we will include a hat and a subscript to denote the metric being used. We will also use the parameters $a$ and $b$ for families of metrics on these spaces.
- In a similar vein, analogous quantities on the singular spaces ($X_0$ and $V_0$) and the smoothings ($X_t$ and $V_t$) will not have a hat, but will include the appropriate subscript. The parameters used for families of metrics here will be $s$ and $t$.
- At times, we will present lemmata and results that can be applied in more general settings, encompassing all of the above spaces. In this case, we will not include the hat, but will use the Greek letters $\alpha$ and $\beta$ as parameters.

For example, we have the following:

| | |
|---|---|
| $\widehat{g}_{\mathrm{co},1}$ | Candelas–de la Ossa metric on the small resolution $\widehat{V}$ at $a = 1$. |
| $g_{\mathrm{co},1}$ | Candelas–de la Ossa metric on the smoothing $V_1$ at $t = 1$. |
| $\widehat{d}_{\mathrm{FLY},a}$ | Distance w.r.t. the balanced metric $\widehat{g}_{\mathrm{FLY},a}$ on the small resolution $\widehat{X}$. |
| $d_{\mathrm{FLY},t}$ | Distance w.r.t. the balanced metric $g_{\mathrm{FLY},t}$ on the smoothing $X_t$. |
| $\widehat{L}_{\widehat{H}_a}(\gamma)$ | Length of a curve $\gamma$ w.r.t. the HYM metric $\widehat{H}_a$ on the small resolution $\widehat{X}$. |
| $L_{H_t}(\gamma)$ | Length of a curve $\gamma$ w.r.t. the HYM metric $H_t$ on the smoothing $X_t$. |
| $\mathrm{diam}_\alpha(Q)$ | Diameter of a set $Q$ w.r.t. a metric $g_\alpha$ on a manifold $X$. |

Table 2.1: Notational Examples for Quantities involving Metrics on Conifold Transitions

In addition, from now on we adopt the convention that $C$ denotes a generic positive constant that may change from line to line but does not depend on $a$ or $t$.

# Chapter 3

# Gromov–Hausdorff Continuity of Conifold Transitions

Since conifold transitions allow us to traverse the moduli space of Calabi–Yau threefolds, it is expected that this process be continuous in some sense. In this chapter, we apply this idea to the Hermitian Yang–Mills metric pairs $(\widehat{\omega}_{\mathrm{FLY},a}, \widehat{H}_a)$ on $\widehat{X}$ and $(\omega_{\mathrm{FLY},t}, H_t)$ on $X_t$ constructed in the previous chapter. Though these are only partial solutions to the Hull–Strominger system (1.22) - (1.25) – one can verify that these metrics fail the heterotic Bianchi identity (1.23) – it is expected that solutions to the full system can be obtained from these via perturbative methods (see §4 for a discussion on one such method).

## 3.1 The Gromov–Hausdorff Topology

The Gromov–Hausdorff topology was introduced by Edwards [Edw75], and was then independently rediscovered by Gromov in the 1980's. Since then, it has been an indispensable tool in geometry. There has been growing interest in applications of the Gromov–Hausdorff topology to Calabi–Yau manifolds starting with the work of Gross–Wilson [GW00]. In particular, it has been applied in the study of the continuity of conifold transitions (see *e.g.,* [RZ11a, RZ11b, Son15]).

We briefly introduce some definitions and notation pertaining to the Gromov–Hausdorff convergence of compact metric spaces. Other sources for this material include *e.g.,* [BBI01, Gro07, GW00, Edw75, Pet06]. We implicitly assume that all metric spaces in this section are compact, though generalizations do exist in the non-compact case ( *cf.,* pointed Gromov–Hausdorff topology/convergence).

Let $(X, d)$ be a compact metric space. For $Q \subseteq X$ and $\epsilon > 0$, we set

$$B_\epsilon(Q) = \cup_{x \in Q} B_\epsilon(x), \tag{3.1}$$

where $B_\epsilon(x) = \{x' \in X \mid d(x, x') < \epsilon\}$ is the ball of radius $\epsilon$ around the point $x$.

**Definition 3.1.1.** Let $(X, d_X)$ and $(Y, d_Y)$ be compact metric spaces and $\epsilon > 0$. A map $f : X \to Y$ is called an $\epsilon$-isometry if

i) $\left|d_X(x, x') - d_Y\big(f(x), f(x')\big)\right| < \epsilon$ for all $x, x' \in X$; and

ii) $Y \subseteq B_\epsilon\big(f(X)\big)$.

In general, $\epsilon$-isometries need not be injective or even continuous.

**Definition 3.1.2.** The Gromov–Hausdorff distance $d_{\mathrm{GH}}$ between two compact metric spaces $(X, d_X)$ and $(Y, d_Y)$ is

$$\begin{aligned} d_{\mathrm{GH}}(X, Y) = \inf\{\epsilon > 0 \mid & \text{ there exist } \epsilon\text{-isometries} \\ & f_1 : X \to Y \text{ and } f_2 : Y \to X\}. \end{aligned} \tag{3.2}$$

The Gromov–Hausdorff distance $d_{\mathrm{GH}}$ defines a metric, and hence a topology, on the set $\mathcal{M}$ of isometry classes of compact metric spaces.

**Remark 3.1.3.** We note that only one of the $\epsilon$-isometries in Definition 3.1.2 is required. This is because given an $\epsilon$-isometry $f_1 : X \to Y$, one can construct a $3\epsilon$-isometry $f_2 : Y \to X$. This, in essence, scales the Gromov–Hausdorff metric $d_{\mathrm{GH}}$ by a factor of 3, however both generate the same topology on $\mathcal{M}$.

## 3.2 The Regular Case

In this section, we will prove several continuity results in the regular case. These regard the small resolution geometries $(\widehat{X}, \widehat{d}_{\mathrm{FLY},a})$ and $(\widehat{X}, \widehat{d}_{\widehat{H}_a})$ for $a > 0$ with respect to the Gromov–Hausdorff topology. We will also prove analogous results for the smoothing geometries $(X_t, d_{\mathrm{FLY},t})$ and $(X_t, d_{H_t})$ for $t \neq 0$.

Our main result is as follows:

**Theorem 3.2.1.** *Let $\widehat{X}$ be a simply connected Kähler Calabi–Yau threefold and let $\widehat{X} \to X_0 \rightsquigarrow X_t$ be a conifold transition. The maps $(0,1] \to \mathcal{M}$ given by*

$$a \mapsto (\widehat{X}, \widehat{d}_{\mathrm{FLY},a}) \text{ and } a \mapsto (\widehat{X}, \widehat{d}_{\widehat{H}_a}) \tag{3.3}$$

*are continuous in the Gromov–Hausdorff topology.*

*Furthermore, the maps $\Delta \setminus \{0\} \to \mathcal{M}$*

$$t \mapsto (X_t, d_{\mathrm{FLY},t}) \text{ and } t \mapsto (X_t, d_{H_t}) \tag{3.4}$$

*are continuous in the Gromov–Hausdorff topology.*

We will extend this result to the intermediate singular spaces $(X_0, d_{\mathrm{FLY},a})$ and $(X_0, d_{H_0})$ in a later section.

### 3.2.1 Gromov–Hausdorff versus Uniform Convergence

Since Riemannian manifolds exhibit more structure than that of a metric space, there is considerably more flexibility when defining notions of continuity than simply using the Gromov–Hausdorff topology. In particular, a very natural way to define continuity of a family of Riemannian manifolds is through some condition on a family of metrics. As the following well-known result shows, the Gromov–Hausdorff topology is weaker than the topology of uniform convergence of Riemannian metrics. Many similar results can be found in the literature ( *cf.*, Example 7.4.4 of [BBI01]).

**Proposition 3.2.2.** *Let $g_\alpha$ be a family of metrics on a connected, compact, manifold $X$ of dimension $n$, where the parameter $\alpha \in U$ lies in a set in either $\mathbb{R}$ or $\mathbb{C}$. Fixing $\beta \in U$, suppose that the map $\alpha \mapsto g_\alpha$ is continuous at $\alpha = \beta$ in the $L^\infty$-norm with respect to $g_\beta$. Then the map $\alpha \mapsto (X, d_\alpha)$ is continuous at $\alpha = \beta$ in the Gromov–Hausdorff topology.*

**Remark 3.2.3.** Since $X$ is compact, all metrics on $X$ are uniformly equivalent. That is, given two metrics $g$ and $\widetilde{g}$ on $X$, there exists some $C > 1$ such that

$$C^{-1} \cdot g \leq \widetilde{g} \leq C \cdot g. \tag{3.5}$$

Thus the continuity assumption in Proposition 3.2.2 could be replaced by the continuity of the family of metrics $g_\alpha$ in the $L^\infty$-norm with respect to any metric on $X$.

*Proof.* Let $0 < \epsilon < 1$ and fix the parameter $\beta \in U$. Consider the identity map $(X, d_\alpha) \to (X, d_\beta)$. This map is surjective, so it suffices to show that it is a $(C \cdot \epsilon)$-isometry when $|\alpha - \beta|$ is small, for some constant $C$ independent of $\alpha$.

The $L^\infty$-continuity of the metrics $g_\alpha$ at $\alpha = \beta$ implies that we may choose $\delta > 0$ sufficiently small such that if $|\alpha - \beta| < \delta$, then $\sup_X \|g_\alpha - g_\beta\|_{g_\beta} < \epsilon$. It follows that there exists some $\epsilon' = \epsilon'(\delta)$ such that for all $\alpha$ with $|\alpha - \beta| < \delta$, we have

$$(1 - \epsilon') \cdot g_\beta \leq g_\alpha \leq (1 + \epsilon') \cdot g_\beta. \tag{3.6}$$

Thus for any $\alpha$ with $|\alpha - \beta| < \delta$, the length of a curve $\gamma$ satisfies $L_\alpha(\gamma) \leq (1 + \epsilon') \cdot L_\beta(\gamma)$. It follows that

$$D := (1 + \epsilon') \cdot \operatorname{diam}_\beta(X) \geq \operatorname{diam}_\alpha(X) \tag{3.7}$$

for all such $\alpha$.

Pick points $p, q \in X$ and choose minimizing geodesics $\gamma_\alpha, \gamma_\beta : [0, 1] \to X$ from $p$ to $q$ in the $g_\alpha$ and $g_\beta$ metrics, respectively. We have that $\gamma_\alpha(0) = \gamma_\beta(0) = p$ and $\gamma_\alpha(1) = \gamma_\beta(1) = q$, and furthermore $L_\alpha(\gamma_\alpha) = d_\alpha(p, q)$ and $L_\beta(\gamma_\beta) = d_\beta(p, q)$. Comparing the lengths of $\gamma_\alpha$ and $\gamma_\beta$ in the metrics $g_\alpha$ and $g_\beta$, we note that

$$\begin{aligned} |L_\alpha(\gamma_\alpha) - L_\beta(\gamma_\alpha)| &\leq \int_0^1 \|g_\beta - g_\beta\|_{g_\beta} \cdot \|\dot{\gamma}_\alpha\|_{g_\beta} \, ds \\ &\leq \left( \sup_X \|g_\alpha - g_\beta\|_{g_\beta} \right) \cdot \int_0^1 \|\dot{\gamma}_\alpha\|_{g_\beta} \, ds \\ &< D \cdot \epsilon. \end{aligned} \tag{3.8}$$

Similarly, we have

$$|L_\alpha(\gamma_\beta) - L_\beta(\gamma_\beta)| < D \cdot \epsilon. \tag{3.9}$$

From this, we see that when $|\alpha - \beta| < \delta$, we have

$$d_\alpha(p, q) \leq L_\alpha(\gamma_\beta) < L_\beta(\gamma_\beta) + D \cdot \epsilon = d_\beta(p, q) + D \cdot \epsilon. \tag{3.10}$$

Similarly,

$$d_\beta(p, q) < d_\alpha(p, q) + D \cdot \epsilon, \tag{3.11}$$

and so

$$|d_\alpha(p, q) - d_\beta(p, q)| < D \cdot \epsilon \text{ when } |\alpha - \beta| < \delta. \tag{3.12}$$

Since this choice of $\delta$ did not depend on the choice of $p, q \in X$, we see that the identity map is a $(D \cdot \epsilon)$-isometry, completing the proof. $\square$

The above can be applied to the metrics on the small resolution $\widehat{X}$, however since the metrics on the smoothings $X_t$ all lie on different spaces, we require a slight variation:

**Corollary 3.2.4.** *Let $(X_\alpha, g_\alpha)$ be a family of connected, compact Riemannian manifolds of dimension $n$, where the parameter $\alpha \in U$ lies in a set in either $\mathbb{R}$ or $\mathbb{C}$. Fixing $\beta \in U$, suppose that for each $\alpha \in U$, there exists a diffeomorphism $F_\alpha : X_\beta \to X_\alpha$ with $F_\beta = \mathrm{Id}_{X_\beta}$. Suppose the map $\alpha \mapsto F_\alpha^*(g_\alpha)$ is continuous at $\alpha = \beta$ in the $L^\infty$-norm with respect to $g_\beta$. Then the map $\alpha \mapsto (X_\alpha, d_\alpha)$ is continuous at $\alpha = \beta$ in the Gromov–Hausdorff topology.*

*Proof.* The proof follows by applying the Proposition 3.2.2 to the metrics $F_\alpha^*(g_\alpha)$ on the fixed manifold $X_\beta$. As such, the map $\alpha \mapsto \big(X_\beta, F_\alpha^*(g_\alpha)\big)$ is continuous at $\alpha = \beta$ in the Gromov–Hausdorff topology. Since $\big(X_\beta, F_\alpha^*(g_\alpha)\big)$ is isometric to $(X_\alpha, g_\alpha)$, we get the desired result. $\square$

### 3.2.2 The Small Resolution Metrics $\widehat{g}_{\mathrm{FLY},a}$

In order to prove Theorem 3.2.1, it suffices by Proposition 3.2.2 to show that each of the families of metrics is continuous in the $L^\infty$-norm. We begin with the Fu–Li–Yau balanced metrics on the small resolution $\widehat{X}$.

**Lemma 3.2.5.** *The Fu–Li–Yau metrics $\widehat{g}_{\mathrm{FLY},a}$ on $\widehat{X}$ satisfy the continuity condition of Proposition 3.2.2 at each $\beta \in U = (0, 1]$.*

*Proof.* Fix $b \in (0, 1]$. Recall that the Fu–Li–Yau metrics are obtained via a gluing construction which interpolates between a multiple of the Candelas–de la Ossa metrics $\widehat{g}_{\mathrm{co},a}$ near the contracted $(-1,-1)$-curves and the ambient Calabi–Yau metric $\widehat{\omega}_{\mathrm{CY}}$ away from the curves [CPY24, FLY12]. The gluing region is independent of the parameter $a$ and $\widehat{\omega}^2_{\mathrm{FLY},a} - \widehat{\omega}^2_{\mathrm{FLY},b}$ is supported on open sets around each $(-1,-1)$-curve. In particular, we have the following expression on the local models with $r^3 < 1$:

$$
\begin{aligned}
&\widehat{\omega}^2_{\mathrm{FLY},a} - \widehat{\omega}^2_{\mathrm{FLY},b} \\
&= C\frac{2R^{-1}}{3}\sqrt{-1}\partial\overline{\partial}\left(\chi\Big(\frac{2R^2}{3}f_a(r^3)\Big)\cdot\Big(\sqrt{-1}\partial\overline{\partial}f_a(r^3) + 8a^2\pi^*\widehat{\omega}_{\mathrm{FS}}\Big)\right) \\
&\quad - C\frac{2R^{-1}}{3}\sqrt{-1}\partial\overline{\partial}\left(\chi\Big(\frac{2R^2}{3}f_b(r^3)\Big)\cdot\Big(\sqrt{-1}\partial\overline{\partial}f_b(r^3) + 8b^2\pi^*\widehat{\omega}_{\mathrm{FS}}\Big)\right),
\end{aligned}
\tag{3.13}
$$

where $C$ and $R$ are constants, $\chi$ is a smooth cutoff function, and $f_a$ and $f_b$ are defined by (2.7). It follows that the map $\|\widehat{\omega}^2_{\mathrm{FLY},a} - \widehat{\omega}^2_{\mathrm{FLY},b}\|^2_{\widehat{g}_{\mathrm{FLY},b}}$ is smooth in $a$ and $p \in \widehat{X}$.

Since $b \neq 0$, we can pick some $h > 0$ such that $I = [b-h, b+h] \subseteq (0,1]$ (or $I = [b-h, 1] \subseteq (0,1]$ if $b = 1$). One can check that in coordinates around a point $p \in \widehat{X}$, each component in (3.13) is smooth in $a$ and $p$. In particular, differentiating $f_a(r^3)$ involves uniform bounds since we have the expression $f_a(r^3) = a^2 \cdot f_1(\frac{r^3}{a^3})$ and because $a > 0$, hence $\frac{r^3}{a^3}$ lies in a compact set.

It follows that the covariant derivative of $\|\widehat{\omega}^2_{\mathrm{FLY},a} - \widehat{\omega}^2_{\mathrm{FLY},b}\|^2_{\widehat{g}_{\mathrm{FLY},b}}$ is continuous on $I \times \widehat{X}$. By compactness, we obtain uniform boundedness of the covariant derivative on $I$. Then, using a corollary of the Arzelà–Ascoli Theorem, the pointwise convergence of the function $\|\widehat{\omega}^2_{\mathrm{FLY},a} - \widehat{\omega}^2_{\mathrm{FLY},b}\|^2_{\widehat{g}_{\mathrm{FLY},b}}$ is actually uniform.

In general, a positive $(n-1, n-1)$-form has a unique $(n-1)$-th root which is determined in a continuous fashion (see *e.g.,* [Mic82]). Applying this, it follows that

$$
\lim_{a \to b} \sup_{\widehat{X}} \|\widehat{g}_{\mathrm{co},a} - \widehat{g}_{\mathrm{co},b}\|_{\widehat{g}_{\mathrm{co},b}} = 0 \tag{3.14}
$$

as desired. □

### 3.2.3 The Smoothing Metrics $g_{\mathrm{FLY},t}$

We now prove the analogue of Lemma 3.2.5 for the smoothing metrics $g_{\mathrm{FLY},t}$ on $X_t$.

Recall from Definition 1.2.4 that a conifold transition involves a holomorphic smoothing $\mu : \mathcal{X} \to \Delta$ of $X_0$ with fibers $X_t = \mu^{-1}(t)$. Fix $s \neq 0$ and consider the smoothings $X_t$ near $X_s$. As this is a smooth family of complex manifolds, Ehresmann's Lemma yields a smoothly varying family of diffeomorphisms $F_t : X_s \to X_t$ such that $F_s = \mathrm{Id}_{X_s}$.

From §2.2.2, the Fu–Li–Yau metrics $g_{\mathrm{FLY},t}$ were constructed in two steps. First, a gluing construction, which resulted in a family of auxiliary metrics $g_{\mathrm{aux},t}$, and then a perturbation.

The expression (2.48) for $\omega^2_{\mathrm{aux},t}$ is smooth in the parameter $t$ and hence we can employ the method in the proof of Lemma 3.2.5. Since the square root construction is continuous and the family of diffeomorphisms $F_t : X_s \to X_t$ varies smoothly, we have

$$\lim_{t\to s} \sup_{X_s} \|F_t^*(g_{\mathrm{aux},t}) - g_{\mathrm{aux},s}\|_{g_{\mathrm{aux},s}} = 0. \tag{3.15}$$

Corollary 3.2.4 then applies to the auxiliary spaces $(X_t, g_{\mathrm{aux},t})$, however these are not the desired balanced metrics. For this, we need to estimate the correction term $\gamma_t$ appearing in (2.49).

The correction term $\gamma_t$ comes from solving

$$E_t(\gamma_t) = \overline{\partial}\omega^2_{\mathrm{aux},t}. \tag{3.16}$$

Hence we need to deduce that the solutions $\gamma_t$ vary smoothly from the fact that the right-hand sides $\overline{\partial}\omega^2_{\mathrm{aux},t}$ vary smoothly for $t \neq 0$. This will follow from properties of the Kodaira–Spencer operator $E_t$ (which here is determined with respect to the auxiliary metrics $\omega_{\mathrm{aux},t}$).

**Lemma 3.2.6.** *Let $\widehat{X}$ be a Kähler Calabi–Yau threefold and let $\widehat{X} \to X_0 \rightsquigarrow X_t$ be a conifold transition. Endow $X_t$ with the auxiliary Hermitian metric $\omega_{\mathrm{aux},t}$ from the Fu–Li–Yau construction [FLY12]. Then $E_t : \Lambda^{2,3}(X_t) \to \Lambda^{2,3}(X_t)$ satisfies* $\ker E_t = \{0\}$ *for all $t$ with $|t|$ sufficiently small.*

*Proof.* Recall that the Kodaira–Spencer operator is defined by

$$E_t = \partial\overline{\partial}\overline{\partial}^\dagger\partial^\dagger + \partial^\dagger\overline{\partial}\overline{\partial}^\dagger\partial + \partial^\dagger\partial. \tag{3.17}$$

Let $\xi \in \Lambda^{2,3}(X_t)$ be such that $\xi \in \ker E_t$. Integrating the identity $\langle E_t(\xi), \xi\rangle = 0$ by parts implies that

$$\partial\xi = 0, \qquad \overline{\partial}^\dagger\partial^\dagger\xi = 0. \tag{3.18}$$

We note that $\partial^\dagger \xi \in \Lambda^{1,3}(X_t)$, and so $\overline{\partial}(\partial^\dagger \xi) = 0$ by type consideration. It is noted in [FLY12] that

$$H^{1,3}(X_t, \mathbb{C}) = H^0(X_t, TX_t) = 0 \tag{3.19}$$

by using that $H^{1,3}(\widehat{X}, \mathbb{C}) = 0$ on the small resolution together with Hartog's Lemma. As such, there exists some $\eta \in \Lambda^{1,2}(X_t)$ such that

$$\partial^\dagger \xi = \overline{\partial} \eta \tag{3.20}$$

and so

$$\langle \partial^\dagger \xi, \partial^\dagger \xi \rangle = \langle \overline{\partial}^\dagger \partial^\dagger \xi, \eta \rangle = 0. \tag{3.21}$$

From this, we can conclude that if $\xi \in \Lambda^{2,3}(X_t) \cap \ker E_t$, then $\partial \xi = \partial^\dagger \xi = 0$. It follows that $\overline{\xi} \in \Lambda^{3,2}(X_t)$ satisfies

$$\Delta_{\overline{\partial}} \overline{\xi} = 0 \text{ where } \Delta_{\overline{\partial}} = \overline{\partial}\overline{\partial}^\dagger + \overline{\partial}^\dagger \overline{\partial}. \tag{3.22}$$

By the Hodge Theorem, this defines an element in Dolbeault cohomology, and since $X_t$ has trivial canonical bundle, we have

$$H^{3,2}(X_t, \mathbb{C}) = H^2(X_t, \Omega^3_{X_t}) = H^2(X_t, \mathcal{O}_{X_t}). \tag{3.23}$$

Lemma 8.2 in [Fri91] states that if $H^2(\widehat{X}, \mathcal{O}_{\widehat{X}}) = 0$, then $H^2(X_t, \mathcal{O}_{X_t}) = 0$. Since

$$H^2(\widehat{X}, \mathcal{O}_{\widehat{X}}) = H^{0,2}(\widehat{X}, \mathbb{C}) = H^{0,1}(\widehat{X}, \mathbb{C}) = 0 \tag{3.24}$$

on the initial Kähler Calabi–Yau threefold $\widehat{X}$, we conclude that $H^{3,2}(X_t, \mathbb{C}) = 0$ and so $\xi = 0$. □

We will also need some uniform estimates at $t \to s$. This is a standard argument given that $\ker E_t$ is trivial and $X_t$ is smooth.

**Lemma 3.2.7.** *Fix $s \neq 0$. There exists $\epsilon > 0$ and $C > 1$ such that*

$$\|\xi_t\|_{C^{4,\alpha}(X_t)} \leq C \cdot \|E_t(\xi_t)\|_{C^\alpha(X_t)} \tag{3.25}$$

*for all $\xi_t \in \Lambda^{2,3}(X_t)$ with $|t - s| < \epsilon$. Here each norm on $X_t$ is taken with respect to the auxiliary Hermitian metrics $\omega_{\text{aux},t}$ from the Fu–Li–Yau construction.*

*Proof.* Since the compact manifolds $X_t$ deform smoothly to the compact manifold $X_s$, the Schauder estimates

$$\|\xi_t\|_{C^{4,\alpha}(X_t)} \leq C \cdot \left(\|\xi_t\|_{C^0(X_t)} + \|E_t(\xi_t)\|_{C^\alpha(X_t)}\right) \tag{3.26}$$

hold uniformly for all $t$ sufficiently close to $s$ where the norms on $X_t$ are taken with respect to $\omega_{\mathrm{aux},t}$. We would like to upgrade this estimate to (3.25).

Suppose that (3.25) is false. Then there exists a sequence $t_j \to s$ and constants $C_j \to \infty$ such that

$$\|\xi_{t_j}\|_{C^{4,\alpha}(X_{t_j})} > C_j \cdot \|E_{t_j}(\xi_{t_j})\|_{C^\alpha(X_{t_j})}. \tag{3.27}$$

Consider the normalized sequence given by

$$\widetilde{\xi}_{t_j} = \frac{\xi_{t_j}}{\|\xi_{t_j}\|_{C^{4,\alpha}(X_{t_j})}}. \tag{3.28}$$

By our assumption, we have

$$\|\widetilde{\xi}_{t_j}\|_{C^{4,\alpha}(X_{t_j})} = 1, \qquad \|E_{t_j}(\widetilde{\xi}_{t_j})\|_{C^\alpha(X_{t_j})} < C_j^{-1}, \tag{3.29}$$

and so we may apply the Arzelà–Ascoli Theorem to extract a convergent subsequence with limit $\widetilde{\xi}_\infty$ satisfying

$$E_s(\widetilde{\xi}_\infty) = 0. \tag{3.30}$$

By the previous lemma $\widetilde{\xi}_\infty = 0$. However, we have (3.26), which says

$$1 \leq C \cdot \left(\|\widetilde{\xi}_{t_j}\|_{C^0(X_{t_j})} + \|E_{t_j}(\widetilde{\xi}_{t_j})\|_{C^\alpha(X_{t_j})}\right), \tag{3.31}$$

and so

$$\frac{1}{2C} \leq \|\widetilde{\xi}_{t_j}\|_{C^0(X_{t_j})} \tag{3.32}$$

for sufficiently large $j$. This means that

$$\|\widetilde{\xi}_\infty\|_{C^0(X_s)} > 0, \tag{3.33}$$

which is a contradiction. □

Returning to the construction of the metrics $\omega_{\mathrm{FLY},t}$, we claim that $F_t^*(\gamma_t) \to \gamma_s$ in $C^4(X_s)$ as $t \to s$. Suppose not, so that there exists $\epsilon > 0$ with

$$\|F_{t_j}^*(\gamma_{t_j}) - \gamma_s\|_{C^4(X_s)} \geq \epsilon \tag{3.34}$$

along a subsequence $t_j \to s$. The uniform elliptic estimate (3.25) implies that $\|\gamma_t\|_{C^{4,\alpha}(X_t)} \leq C$, and so $F_t^*(\gamma_t)$ is also bounded on $(X_s, g_{\mathrm{aux},s})$. Applying the Arzelà–Ascoli Theorem, there is a subsequence converging to a limit $\gamma_\infty$ on $X_s$ solving

$$E_s(\gamma_\infty) = \overline{\partial}\omega^2_{\mathrm{aux},s}. \tag{3.35}$$

It follows that

$$E_s(\gamma_\infty - \gamma_s) = 0, \tag{3.36}$$

and since $\ker E_s = \{0\}$, we conclude that $\gamma_\infty = \gamma_s$, which contradicts (3.34).

Using that $F_t^*(\gamma_t) \to \gamma_s$, taking a square root of (2.49) gives a family of metrics $\omega_{\mathrm{FLY},t}$ that vary continuously. Hence

$$\lim_{t\to s} \sup_{X_s} \|F_t^*(g_{\mathrm{FLY},t}) - g_{\mathrm{FLY},s}\|_{g_{\mathrm{aux},s}} = 0. \tag{3.37}$$

By Remark 3.2.3, this convergence also holds with respect to the Fu–Li–Yau metric $g_{\mathrm{FLY},s}$ and thus Corollary 3.2.4 applies to the family $(X_t, g_{\mathrm{FLY},t})$. This shows that $(X_t, d_{\mathrm{FLY},t}) \to (X_s, d_{\mathrm{FLY},s})$ in the Gromov–Hausdorff topology as $t \to s$.

### 3.2.4 The Small Resolution Metrics $\widehat{H}_a$

We again consider the small resolution $\widehat{X}$, where there is a family of metrics $\widehat{H}_a$ satisfying the Hermitian Yang–Mills equation

$$F_{\widehat{H}_a} \wedge \widehat{\omega}^2_{\mathrm{FLY},a} = 0. \tag{3.38}$$

We will show that for fixed $b \in (0,1]$,

$$\lim_{a\to b} \sup_{\widehat{X}} \|\widehat{H}_a - \widehat{H}_b\|_{\widehat{H}_b} = 0. \tag{3.39}$$

Suppose instead that this is false. Then there exists $\epsilon > 0$ and a sequence $a_j \to b$ such that

$$\|\widehat{H}_{a_j} - \widehat{H}_b\|_{\widehat{H}_b} \geq \epsilon, \qquad \sqrt{-1}\Lambda_{\widehat{\omega}_{\mathrm{FLY},a_j}} F_{\widehat{H}_{a_j}} = 0 \tag{3.40}$$

for all $a_j$. By the estimates in Proposition 2.1.1 and uniform equivalence of the Fu–Li–Yau metrics $\widehat{g}_{\mathrm{FLY},a}$ near $b$, we have that

$$C^{-1} \cdot \widehat{g}_{\mathrm{FLY},b} \leq \widehat{H}_a \leq C \cdot \widehat{g}_{\mathrm{FLY},b}. \tag{3.41}$$

Also, standard estimates for the Hermitian Yang–Mills equations give that

$$\|\nabla \widehat{H}_a\|_{\widehat{g}_{\mathrm{FLY},b}} + \|\nabla^2 \widehat{H}_a\|_{\widehat{g}_{\mathrm{FLY},b}} \leq C. \tag{3.42}$$

(See *e.g.,* Proposition 3.9 of [CPY24] with the function $r \equiv 1$ and the higher-order estimates that follow for a proof of these standard estimates.)

By the Arzelà–Ascoli Theorem, we may extract a subsequence of $\widehat{H}_{a_j}$ that converges to a limit $\widehat{H}_\infty$ such that

$$\|\widehat{H}_\infty - \widehat{H}_b\|_{\widehat{H}_b} \geq \epsilon, \qquad i\Lambda_{\widehat{\omega}_{\mathrm{FLY},b}} F_{\widehat{H}_\infty} = 0, \tag{3.43}$$

where we have use the fact that $\widehat{\omega}_{\mathrm{FLY},a_j} \to \widehat{\omega}_{\mathrm{FLY},b}$ as $a_j \to b$.

We now have two Hermitian Yang–Mills metrics $\widehat{H}_\infty$ and $\widehat{H}_b$ with respect to $\widehat{\omega}_{\mathrm{FLY},b}$. By the uniqueness of Hermitian Yang–Mills metrics (see *e.g.,* [Don85]), we have that these must be multiples of one another, that is $\widehat{H}_\infty = \lambda \cdot \widehat{H}_b$. The normalization condition (2.25), however, tells us that $\lambda = 1$, which contradicts (3.43).

As such, we have that (3.39) holds and we conclude that $(\widehat{X}, \widehat{d}_{\widehat{H}_a}) \to (\widehat{X}, \widehat{d}_{\widehat{H}_b})$ in the Gromov–Hausdorff sense as $a \to b$.

### 3.2.5 The Smoothing Metrics $H_t$

We now work with the final set of metrics $H_t$ on $X_t$. Fix $s \neq 0$ and consider the smoothings $X_t$ near the smooth fiber $X_s$ with smoothly varying family of diffeomorphisms $F_t : X_s \to X_t$ with $F_s = \mathrm{Id}_{X_s}$. As in the previous sections, we show that the metrics $H_t$ also satisfy continuity of the form

$$\lim_{t \to s} \sup_{X_s} \|F_t^*(H_t) - H_s\|_{H_s} = 0. \tag{3.44}$$

The proof of this is similar to our earlier arguments. First, suppose otherwise and extract a converging subsequence $H_{t_j}$ with $t_j \to s$ via the estimates (2.56) and (3.42). The limit $H_\infty$ solves the Hermitian Yang–Mills equation with respect to $g_{\mathrm{FLY},s}$ and hence by uniqueness and normalization, must be equal to $H_s$, which is a contradiction.

As such, by applying Corollary 3.2.4 we get that $(X_t, d_{H_t}) \to (X_s, d_{H_s})$ in the Gromov–Hausdorff topology as $t \to s$.

## 3.3 The Singular Case

The previous section (§3.2) was concerned with the continuity of the families of Riemannian manifolds on either side of a conifold transition $\widehat{X} \to X_0 \rightsquigarrow X_t$. In this chapter, we extend these results to include the intermediate singular spaces, which we can consider as complete metric spaces by the process described in §2.3.

The main result of this chapter is the following:

**Theorem 3.3.1.** *Let $\widehat{X}$ be a simply connected Kähler Calabi–Yau three-fold and let $\widehat{X} \to X_0 \rightsquigarrow X_t$ be a conifold transition. The following four convergences hold in the Gromov–Hausdorff topology:*

$$\begin{array}{cc} \underline{\textit{As } a \to 0\textit{:}} & \underline{\textit{As } t \to 0\textit{:}} \\ (\widehat{X}, \widehat{d}_{\mathrm{FLY},a}) \to (X_0, d_{\mathrm{FLY},0}), & (X_t, d_{\mathrm{FLY},t}) \to (X_0, d_{\mathrm{FLY},0}), \\ (\widehat{X}, \widehat{d}_{\widehat{H}_a}) \to (X_0, d_{H_0}), & (X_t, d_{H_t}) \to (X_0, d_{H_0}). \end{array}$$

*Therefore the maps $[0,1] \to \mathcal{M}$ given by*

$$a \mapsto (\widehat{X}, \widehat{d}_{\mathrm{FLY},a}) \text{ and } a \mapsto (\widehat{X}, \widehat{d}_{\widehat{H}_a}), \tag{3.45}$$

*and the maps $\Delta \to \mathcal{M}$*

$$t \mapsto (X_t, d_{\mathrm{FLY},t}) \text{ and } t \mapsto (X_t, d_{H_t}) \tag{3.46}$$

*are continuous and agree at $a = t = 0$.*

The simply connected and Kähler conditions are required here since they were needed for the metric constructions of Fu–Li–Yau and Collins–Picard–Yau in §2.

In proving the above, we will make use of the following theorem (Theorem 2.5.23 of [BBI01]):

**Theorem 3.3.2.** *Let $(X, d)$ be a complete, locally compact length space. Then given $p, q \in X$, there exists an admissible curve $\gamma : [0,1] \to X$ such that $\gamma(0) = p$ and $\gamma(1) = q$ with $L(\gamma) = d(p,q)$.*

In the above, admissible curves are a subset of continuous curves in $X$ that are closed under restrictions, concatenations, and linear reparameterizations. While the upcoming definitions and results hold in the more general length space setting, we keep in mind that for our purposes, we will later take these to be piecewise smooth curves on our manifolds.

We will adopt the convention that the diameter of a set $Q$ refers to the intrinsic diameter, as defined below:

**Definition 3.3.3.** Let $Q$ be a bounded, path connected set in a length space $X$. Given two points $p, q \in Q$, the intrinsic distance $d_{\text{int}}(p, q)$ between $p$ and $q$ is defined by

$$d_{\text{int}}(p, q) := \inf_{\gamma} L(\gamma), \tag{3.47}$$

where the infimum is taken over all admissible curves $\gamma$ from $p$ to $q$ contained in $Q$.

The (intrinsic) diameter of $Q$ is defined by

$$\operatorname{diam}(Q) := \sup_{p,q \in Q} d_{\text{int}}(p, q). \tag{3.48}$$

We note that this is a non-standard definition, since many other authors take the diameter of $Q$ to be the supremum of the distance (in $X$) between pairs of points in $Q$.

### 3.3.1 Reduction of Curves

Recall that we have defined the distance between points on our spaces by integrating over admissible curves and taking infima. As such, it is helpful to consider curves that avoid pathological behaviour. For our purposes, we have the following Curve Reduction Lemma, which allows us to work with curves that only enter each of a finite collection "bad" sets at most once.

**Lemma 3.3.4.** *Let $Q_1, \dots, Q_k$ be disjoint, closed, path-connected, bounded sets in a complete, locally compact length space $(X, d)$, and let $\gamma : [0, 1] \to X$ be an admissible curve. Then there exists an admissible curve $\mu : [0, 1] \to X$ such that*

*i)* $\mu(0) = \gamma(0)$ *and* $\mu(1) = \gamma(1)$*;*

*ii) for all* $j \in \{1, \dots, k\}$*, the set* $\mu^{-1}(Q_j) \subseteq [0, 1]$ *is either empty of a single closed subinterval of* $[0, 1]$*; and*

*iii) we have the estimate (noting Definition 3.3.3)*

$$L(\mu) \leq L(\gamma) + \sum_{j=1}^{k} \operatorname{diam}(Q_j). \tag{3.49}$$

*Proof.* We construct the curve $\mu$ in the following manner:

Define $a_1 \in [0,1]$ by

$$a_1 := \inf\{s \in [0,1] \mid \gamma(s) \in \cup_{j=1}^{k} Q_j\}. \tag{3.50}$$

Relabelling the sets $Q_j$ if necessary, we can say that $\gamma(a_1) \in Q_1$. Now, define a time $b_1 \in [0,1]$ by

$$b_1 := \sup\{s \in [0,1] \mid \gamma(s) \in Q_1\}. \tag{3.51}$$

Using Theorem 3.3.2, we can take $\mu|_{[a_1,b_1]}$ to be any admissible curve such that $\mu([a_1,b_1]) \subseteq Q_1$, the endpoints $\mu(a_1) = \gamma(a_1)$ and $\mu(b_1) = \gamma(b_1)$ match, and also

$$L\big(\mu|_{[a_1,b_1]}\big) \le \operatorname{diam}(Q_1). \tag{3.52}$$

For $j > 1$, we can define $a_j$ by

$$a_j := \inf\{s \in (b_{j-1},1] \mid \gamma(s) \in \cup_{j=1}^{k} Q_j\}, \tag{3.53}$$

and relabel the sets such that $\gamma(a_j) \in Q_j \neq Q_1, \ldots, Q_{j-1}$. Define $b_j$ by

$$b_j := \sup\{s \in [0,1] \mid \gamma(s) \in Q_j\} \tag{3.54}$$

and once again choose $\mu|_{[a_j,b_j]}$ to be any admissible curve such that $\mu([a_j,b_j]) \subseteq Q_j$, the endpoints $\mu(a_j) = \gamma(a_j)$ and $\mu(b_j) = \gamma(b_j)$ match, and also

$$L\big(\mu|_{[a_j,b_j]}\big) \le \operatorname{diam}(Q_j). \tag{3.55}$$

Eventually, after $\ell \le k$ iterations, we will not have any $a_{\ell+1}$.

At this point, we have constructed the curve $\mu$ on the set $A = \cup_{j=1}^{\ell}[a_j,b_j]$. To finish the curve, we set $\mu(s) = \gamma(s)$ for $s \in A' = [0,1] \setminus A$.

Since the class of admissible curves is closed under restrictions and concatenations (see Definition 2.1.1 of [BBI01]), we see by construction the $\mu$ is admissible. Furthermore, $\mu^{-1}(Q_j) = [a_j,b_j]$ for $1 \le j \le \ell$, and $\mu^{-1}(Q_j) = \emptyset$ otherwise. Finally, we note that

$$\begin{aligned} L(\mu) = L\big(\mu|_{A'}\big) + \sum_{j=1}^{\ell} L\big(\mu|_{[a_j,b_j]}\big) &\le L\big(\gamma|_{A'}\big) + \sum_{j=1}^{\ell} \operatorname{diam}(Q_j) \\ &\le L(\gamma) + \sum_{j=1}^{k} \operatorname{diam}(Q_j). \end{aligned} \tag{3.56}$$

□

### 3.3.2 The Main Lemma

To obtain the desired Gromov–Hausdorff convergence, we first prove a general lemma, which encompasses the metrics on both the small resolution $\widehat{X}$ and the smoothings $X_t$. This method is similar to the one used in [SW13a]. With this lemma in place, it only remains to verify its hypothesis in each of our geometric setups.

**Lemma 3.3.5.** *Let $X_\alpha$ be a family of connected compact smooth manifolds where the parameter $\alpha$ lies in either $(0,1] \subseteq \mathbb{R}$ or $\Delta \setminus \{0\} \subseteq \mathbb{C}$. Let $X_0$ be a compact analytic space with $X_0 = (X_0)_{\mathrm{reg}} \cup (X_0)_{\mathrm{sing}}$ where $(X_0)_{\mathrm{reg}}$ is a connected smooth manifold and $(X_0)_{\mathrm{sing}}$ consists of finitely many ODP singular points $\{s_1, \ldots, s_k\}$, meaning that each $s_j \in X_0$ is contained a neighbourhood $U_j \subseteq X_0$ which can be identified with a neighbourhood of the origin in $V_0 \subseteq \mathbb{C}^4$.*

*For each $\alpha$, let $K_{j,\alpha} \subseteq X_0$ and $C_{j,\alpha} \subseteq X_\alpha$ be disjoint compact sets with $s_j \in K_{j,\alpha}$ for $j \in \{1, \ldots, k\}$. Suppose further that we have a family of maps $F_\alpha : X_\alpha \to X_0$ such that*

- *i) the restriction $F_\alpha : X_\alpha \setminus \cup_{j=1}^k C_{j,\alpha} \to X_0 \setminus \cup_{j=1}^k K_{j,\alpha}$ is a diffeomorphism; and*
- *ii) for each $j \in \{1, \ldots, k\}$, we have $F_\alpha(C_{j,\alpha}) \subseteq K_{j,\alpha}$.*

*Let $g_\alpha$ be a Riemannian metric on $X_\alpha$ for each $\alpha$. Let $g_0$ be a smooth Riemannian metric on $(X_0)_{\mathrm{reg}}$ satisfying the bound $g_0 \leq C \cdot (dr \otimes dr + r^2 \cdot g_L)$ in a neighbourhood $U_j$ of the singular points $s_j$ and let $d_0$ be the distance function induced by $g_0$ on $X_0$ (see §2.3).*

*Now, let $\epsilon > 0$ and suppose that there exist disjoint open sets $G_1, \ldots, G_k \subseteq X_0$ and $\alpha_0 > 0$ such that each $G_j$ satisfies*

- *i) $K_{j,\alpha} \subseteq G_j$ when $|\alpha| < \alpha_0$;*
- *ii) $(F_\alpha^{-1})^*(g_\alpha)$ converges uniformly to $g_0$ on the compact set $X_0 \setminus \cup_j G_j$ as $\alpha \to 0$;*
- *iii) $\mathrm{diam}_0(G_j) < \epsilon$; and*
- *iv) $\mathrm{diam}_\alpha\big(F_\alpha^{-1}(G_j)\big) < \epsilon$ when $|\alpha| < \alpha_0$.*

*Then there exists $\alpha_1 > 0$ and a constant $C > 0$ independent of $\alpha$ such that*

$$F_\alpha : (X_\alpha, d_\alpha) \to (X_0, d_0) \tag{3.57}$$

*is a* $(C \cdot \epsilon)$*-isometry for all* $\alpha$ *with* $|\alpha| < \alpha_1$*.*

*Proof.* Let $\epsilon > 0$. We first prove that the image of each $F_\alpha$ is $\epsilon$-dense in $X_0$. By our assumptions, the only points of $X_0$ not in $F_\alpha(X_\alpha)$ must lie in some $K_{j,\alpha}$. For each $j$, we can choose some $p \in \overline{G_j} \setminus K_{j,\alpha}$, which is in the image of $F_\alpha$. Since $\mathrm{diam}_0(\overline{G_j}) < \epsilon$, we have that $F_\alpha(X_\alpha)$ is $\epsilon$-dense in $X_0$ with respect to $d_0$ for sufficiently small $\alpha$.

It remains to prove that there exists some $C, \alpha_1 > 0$ such that for all $\alpha$ with $|\alpha| < \alpha_1$,

$$\big| d_\alpha(p,q) - d_0\big(F_\alpha(p), F_\alpha(q)\big)\big| < C \cdot \epsilon \tag{3.58}$$

for each $p, q \in X_\alpha$.

Let $p, q \in X_\alpha$. Using Theorem 3.3.2, pick a curve $\gamma : [0,1] \to X_0$ such that $\gamma(0) = F_\alpha(p)$ and $\gamma(1) = F_\alpha(q)$ and

$$L_0(\gamma) = d_0\big(F_\alpha(p), F_\alpha(q)\big). \tag{3.59}$$

We will replace this curve $\gamma$ with a new curve $\mu$ on $X_0$ passing through the "bad" sets $\overline{G_j}$ at most $k$ times using the Curve Reduction Lemma (Lemma 3.3.4). The new curve $\mu$ is piecewise differentiable with $\mu(0) = F_\alpha(p)$ and $\mu(1) = F_\alpha(q)$ with

$$L_0(\mu) \leq L_0(\gamma) + \sum_{j=1}^{k} \mathrm{diam}_0(\overline{G_j}) \leq L_0(\gamma) + k \cdot \epsilon. \tag{3.60}$$

The construction of Lemma 3.3.4 provides an integer $\ell \leq k$ and a sequence

$$0 \leq a_1 \leq b_1 < \ldots < a_\ell \leq b_\ell \leq 1, \tag{3.61}$$

such that (by relabelling the $s_j$ if necessary) we have $\mu^{-1}(\overline{G_j}) = [a_j, b_j]$ for $1 \leq j \leq \ell$ and $\mu^{-1}(\overline{G_j}) = \emptyset$ for $\ell + 1 \leq j \leq k$. Set $A_j = [a_j, b_j]$ and $A' = [0,1] \setminus \cup_{j=1}^{\ell} A_j$.

Over the closed time intervals $\overline{A'}$, the curve $\mu$ does not enter any $K_{j,\alpha}$, and can be identified with a curve on $X_\alpha$ by the diffeomorphism $F_\alpha$. We use this idea to define a curve $\mu_\alpha : [0,1] \to X_\alpha$. For $s \in \overline{A'}$, we set $\mu_\alpha(s) = F_\alpha^{-1} \circ \mu(s)$, whereas on each $A_j = [a_j, b_j]$, we pick an arbitrary admissible curve with endpoints $\mu_\alpha(a_j) = F_\alpha^{-1} \circ \mu(a_j)$ and $\mu_\alpha(b_j) = F_\alpha^{-1} \circ \mu(b_j)$.

By the triangle inequality and the diameter estimate $\operatorname{diam}_\alpha\big(F_\alpha^{-1}(\overline{G_j})\big) < \epsilon$, we have that

$$\begin{aligned}
d_\alpha(p,q) \le d_\alpha\big(p,\mu_\alpha(a_1)\big) &+ \sum_{j=1}^{\ell} d_\alpha\big(\mu_\alpha(a_j),\mu_\alpha(b_j)\big) \\
&+ \sum_{j=2}^{\ell} d_\alpha\big(\mu_\alpha(b_{j-1}),\mu_\alpha(a_j)\big) + d_\alpha\big(\mu_\alpha(b_\ell),q\big) \\
&\le L_\alpha\big(\mu_\alpha|_{[0,a_1]}\big) + \sum_{j=2}^{\ell} L_\alpha\big(\mu_\alpha|_{[b_{j-1},a_j]}\big) \\
&+ L_\alpha\big(\mu_\alpha|_{[b_\ell,1]}\big) + k\cdot\epsilon \\
&\le \int_{\overline{A'}} \|\dot{\mu_\alpha}(s)\|_{g_\alpha}\,ds + k\cdot\epsilon \\
&= \int_{\overline{A'}} \|\dot{\mu}(s)\|_{(F_\alpha^{-1})^*(g_\alpha)}\,ds + k\cdot\epsilon. \qquad (3.62)
\end{aligned}$$

The set $\overline{A'}$ is defined such that $\mu|_{\overline{A'}}$ lies in $X_0 \setminus \cup_{j=1}^k G_j$. Since the metrics $(F_\alpha^{-1})^*(g_\alpha)$ converge uniformly to $g_0$ on this region, we have that

$$\begin{aligned}
\int_{\overline{A'}} \|\dot{\mu}(s)\|_{(F_\alpha^{-1})^*(g_\alpha)}\,ds &\le (1+\delta)\cdot\int_{\overline{A'}} \|\dot{\mu}(s)\|_{g_0}\,ds \\
&\le (1+\delta)\cdot L_0(\mu), \qquad (3.63)
\end{aligned}$$

where $\delta$ can be made arbitrarily small if $|\alpha|$ is sufficiently small. Applying (3.59) and (3.60) yields

$$\int_{\overline{A'}} \|\dot{\mu}(s)\|_{(F_\alpha^{-1})^*(g_\alpha)}\,ds \le L_0(\mu) + \delta\cdot\big(\operatorname{diam}_0(X_0) + k\cdot\epsilon\big). \qquad (3.64)$$

We note that $\operatorname{diam}_0(X_0) < \infty$ since it is a union of a smooth geometry on a compact manifold $X_0 \setminus \cup_{j=1}^k G_j$ with sets $\overline{G_j}$ of bounded diameter that have non-trivial intersection with $X_0 \setminus \cup_{j=1}^k G_j$. Using (3.62) and (3.63), and choosing $\delta$ small enough yields

$$d_\alpha(p,q) \le L_0(\mu) + (k+1)\cdot\epsilon, \qquad (3.65)$$

which combined with (3.59) and (3.60) tells us that

$$d_\alpha(p,q) \le d_0\big(F_\alpha(p),F_\alpha(q)\big) + (2k+1)\cdot\epsilon. \qquad (3.66)$$

We are left with showing the other side of the inequality (3.58). This is achieved using a similar argument. Let $\eta_\alpha : [0,1] \to X_\alpha$ be a curve such that $\eta_\alpha(0) = p$, $\eta_\alpha(1) = q$, and

$$L_\alpha(\eta_\alpha) = d_\alpha(p, q). \tag{3.67}$$

As before, we use the Curve Reduction Lemma (Lemma 3.3.4) to replace $\eta_\alpha$ with a curve $\nu_\alpha$ that passes through the "bad" sets $F_\alpha^{-1}(\overline{G_j})$ at most $k$ times. The replacement curve $\nu_\alpha : [0,1] \to X_\alpha$ has the same endpoints as $\eta_\alpha$: $\nu_\alpha(0) = p$, $\nu_\alpha(1) = q$, and also satisfies the length estimate

$$L_\alpha(\nu_\alpha) \le L_\alpha(\eta_\alpha) + \sum_{j=1}^{k} \mathrm{diam}_\alpha\big(F_\alpha^{-1}(\overline{G_j})\big) \le L_\alpha(\eta_\alpha) + k \cdot \epsilon. \tag{3.68}$$

The time interval $[0,1]$ can be broken into $[0,1] = A_\alpha \cup A'_\alpha$ as before. Here $A_\alpha$ is the union of closed intervals where the curve enters the $F_\alpha^{-1}(\overline{G_j})$ and the remainder $A'_\alpha$ is such that $\nu_\alpha|_{\overline{A'_\alpha}}$ lies in $X_\alpha \setminus \cup_{j=1}^k F_\alpha^{-1}(G_j)$.

We use the map $F_\alpha$ to identify $\nu_\alpha|_{\overline{A'_\alpha}}$ with a curve $\mu$ on $X_0$ by setting $\nu(s) = F_\alpha \circ \nu_\alpha(s)$. The curve can be extended to all of $[0,1]$ by picking admissible curves with appropriate matching endpoints. Using the triangle inequality and diameter estimate as in (3.62), we get

$$d_0\big(F_\alpha(p), F_\alpha(q)\big) \le \int_{\overline{A'_\alpha}} \|\dot{\nu}(s)\|_{g_0} + k \cdot \epsilon. \tag{3.69}$$

We again use the uniform convergence of the metrics $(F_\alpha^{-1})^*(g_\alpha)$ to $g_0$ on $X_0 \setminus \cup_{j=1}^k G_j$ and the fact that $\nu|_{\overline{A'_\alpha}}$ lies in $X_0 \setminus \cup_{j=1}^k G_j$ to get that

$$\begin{aligned} \int_{\overline{A'_\alpha}} \|\dot{\nu}(s)\|_{g_0} &\le (1+\delta) \cdot \int_{\overline{A'_\alpha}} \|\dot{\nu}(s)\|_{(F_\alpha^{-1})^*(g_\alpha)} \\ &= (1+\delta) \cdot \int_{\overline{A'_\alpha}} \|\dot{\nu_\alpha}(s)\|_{g_\alpha} \\ &\le (1+\delta) \cdot L_\alpha(\nu_\alpha) \\ &\le L_\alpha(\nu_\alpha) + \delta \cdot \big(\mathrm{diam}_\alpha(X_\alpha) + k \cdot \epsilon\big), \end{aligned} \tag{3.70}$$

where $\delta$ is when $|\alpha|$ is sufficiently small.

We can uniformly bound the diameter $\mathrm{diam}_\alpha(X_\alpha)$ when $|\alpha|$ is sufficiently small. To see this, note that $\big(X_\alpha \setminus \cup_{j=1}^k F_\alpha^{-1}(G_j), g_\alpha\big)$ is isometric to $\big(X_0 \setminus$

$\cup_{j=1}^{k} G_i, (F_\alpha^{-1})^*(g_\alpha)\big)$, which has uniformly bounded diameter since the metrics $(F_\alpha^{-1})^*(g_\alpha) \to g_0$ uniformly in this region. The remaining pieces of the geometry $(X_\alpha, g_\alpha)$, namely the sets $F_\alpha^{-1}(\overline{G_j})$, all also have bounded diameter and have non-trivial intersection with $\big(X_\alpha \setminus \cup_{j=1}^{k} F_\alpha^{-1}(G_j), g_\alpha\big)$.

Combining the previous estimates and choosing $\delta$ small enough then yields

$$d_0\big(F_\alpha(p), F_\alpha(q)\big) \leq d_\alpha(p,q) + (2k+1)\cdot\epsilon, \tag{3.71}$$

which together with (3.66) yields the desired (3.58) with uniform constant $C = 2k+1$. □

### 3.3.3 Estimates on the Small Resolution

We now show how the Main Lemma (Lemma 3.3.5) gives the Gromov–Hausdorff convergence of the families of metrics on the small resolution $\widehat{X}$ of a conifold transition $\widehat{X} \to X_0 \rightsquigarrow X_t$. In this case, the manifolds $X_\alpha$ are all taken to be the initial Kähler Calabi–Yau manifold $\widehat{X}$ and the maps $F_\alpha$ are all the blowdown map $\pi : \widehat{X} \to X_0$. The sets $C_{j,\alpha} \subseteq \widehat{X}$ are the contracted $(-1,-1)$-curves $E_j \simeq \mathbb{P}^1$, and the sets $K_{j,\alpha} \subseteq X_0$ are the singleton sets consisting of the conifold singularities $s_j$.

To apply Lemma 3.3.5, we must first find the open sets $G_j$ and verify the uniform diameter estimates in its hypothesis. Since these are local estimates around the $(-1,-1)$-curves and around the singularities, we can work on the local models $(\widehat{V}, \widehat{g}_{\mathrm{co},a})$ and $(V_0, g_{\mathrm{co},0})$. In particular, we will work on "tubular" neighbourhoods

$$\widehat{T}(R) := \{r \leq R\} \subseteq \widehat{V}, \tag{3.72}$$

and discs

$$D_0(R) := \{r \leq R\} \subseteq V_0, \tag{3.73}$$

where $r$ denotes the respective radius functions defined in §2. The sets $\widehat{T}(R)$ and $D_0(R)$ are identified with each other using the blowdown map $\pi$.

Using the Asymptotically Conical Decay Property (CO SR II), we can fix a constant $K > 1$ such that

$$\|(\pi^{-1})^*(\widehat{g}_{\mathrm{co},a}) - g_{\mathrm{co},0}\|_{g_{\mathrm{co},0}} \leq \frac{1}{2} \tag{3.74}$$

when $r > aK$. For $\delta > 0$ and $a \in (0, \frac{\delta}{K})$, it is helpful to split the "tube" $\widehat{T}(\delta)$ into two pieces

$$\widehat{T}(\delta) = \widehat{T}(aK) \cup \Big(\widehat{T}(\delta) \setminus \widehat{T}(aK)\Big). \tag{3.75}$$

We will use the Scaling Property (CO SR I) on the smaller "tube" $\widehat{T}(aK)$, an the Asymptotically Conical Decay Property (CO SR II) on the "annulus" $\widehat{T}(\delta) \setminus \widehat{T}(aK)$.

### "Tubular" Bounds

We begin by obtaining uniform bounds on the space $\big(\widehat{T}(aK), \widehat{g}_{\mathrm{co},a}\big)$. The Scaling Property (CO SR I) says that

$$\widehat{g}_{\mathrm{co},a} = a^2 \cdot S^*_{a^{-1}}(\widehat{g}_{\mathrm{co},1}) \tag{3.76}$$

and so we have

$$S_{a^{-1}} : \big(\widehat{T}(aK), \widehat{g}_{\mathrm{co},a}\big) \to \big(\widehat{T}(K), a^2 \cdot \widehat{g}_{\mathrm{co},1}\big) \tag{3.77}$$

is an isometry. We then get the diameter relation

$$\widehat{\mathrm{diam}}_{\mathrm{co},a}\big(\widehat{T}(aK)\big) = a \cdot \widehat{\mathrm{diam}}_{\mathrm{co},1}\big(\widehat{T}(K)\big). \tag{3.78}$$

We note that this diameter is finite since the set is compact and path connected (there is a deformation retract from $\widehat{T}(K)$ to the path connected zero section $E \simeq \mathbb{P}^1$.

### "Annular" Bounds

We will make use of a convenient class of paths that move along the fibers of the bundle $\widehat{V}$. Fix a point $p = (\lambda_0, u_0, v_0) \in \widehat{T}(\delta) \setminus \widehat{T}(aK)$ and let $\rho = r(p)$. By our choice of $p$, we have $\rho \in (aK, \delta]$. Consider the curve $\widehat{\gamma} : [\frac{aK}{\rho}, 1] \to \widehat{T}(\delta)$ given by

$$\widehat{\gamma}(s) = (\lambda_0, s^{\frac{3}{2}} \cdot u_0, s^{\frac{3}{2}} \cdot v_0). \tag{3.79}$$

This curve begins in $\widehat{T}(aK)$ and moves along the fiber over $\lambda_0$ to arrive at $\widehat{\gamma}(1) = p$.

Using the blowdown map $\pi : \widehat{V} \to V_0$ given by (1.8), one can check that this curve is sent to the curve $\gamma = \pi \circ \widehat{\gamma}$ in $V_0$ given by

$$\gamma(s) = s^{\frac{3}{2}} \cdot \pi(p). \tag{3.80}$$

It follows that

$$r\big(\gamma(s)\big) = s \cdot \rho. \tag{3.81}$$

We require a couple of properties of this curve.

**Lemma 3.3.6.** *The path $\gamma(s)$ on $V_0$ defined above has speed $\|\dot{\gamma}\|_{g_{\mathrm{co},0}} = \rho$ and length $L_{\mathrm{co},0}(\gamma) = \rho - aK$.*

*Proof.* The cone metric can be written as $g_{\mathrm{co},0} = dr \otimes dr + r^2 \cdot g_L$, where $g_L$ is the pullback of a metric on the link $\{r = 1\}$. Let $\mathrm{pr} : V_0 \to L$ denote the projection to the link given by

$$\mathrm{pr}(z) = \frac{z}{\|z\|}. \tag{3.82}$$

We can then compute that

$$dr(\dot{\gamma}) = \frac{d}{ds}(r \circ \gamma) = \rho \tag{3.83}$$

and

$$\mathrm{pr}_*(\dot{\gamma}) = \frac{d}{ds}(\mathrm{pr} \circ \gamma) = 0. \tag{3.84}$$

It follows that

$$g_{\mathrm{co},0}(\dot{\gamma}, \dot{\gamma}) = \rho^2. \tag{3.85}$$

Taking the square root of this gives the speed $\rho$, and integrating the speed over the interval $[\frac{aK}{\rho}, 1]$ yields the length. □

We now compare the length of the curve $(\widehat{\gamma}, \widehat{g}_{\mathrm{co},a})$ to that of the curve $(\gamma, g_{\mathrm{co},0})$.

$$\begin{aligned} |\widehat{L}_{\mathrm{co},a}(\widehat{\gamma}) - L_{\mathrm{co},0}(\gamma)| &= \left| \int_{\frac{aK}{\rho}}^{1} \left( \|\dot{\widehat{\gamma}}\|_{\widehat{g}_{\mathrm{co},a}} - \|\dot{\gamma}\|_{g_{\mathrm{co},0}} \right) ds \right| \\ &\leq \int_{\frac{aK}{\rho}}^{1} \left| \|\dot{\widehat{\gamma}}\|_{\widehat{g}_{\mathrm{co},a}} - \|\dot{\gamma}\|_{g_{\mathrm{co},0}} \right| ds \\ &= \int_{\frac{aK}{\rho}}^{1} \left| \|\dot{\gamma}\|_{(\pi^{-1})^*(\widehat{g}_{\mathrm{co},a})} - \|\dot{\gamma}\|_{g_{\mathrm{co},0}} \right| ds \\ &= \int_{\frac{aK}{\rho}}^{1} \left| \frac{\|\dot{\gamma}\|^2_{(\pi^{-1})^*(\widehat{g}_{\mathrm{co},a})} - \|\dot{\gamma}\|^2_{g_{\mathrm{co},0}}}{\|\dot{\gamma}\|_{(\pi^{-1})^*(\widehat{g}_{\mathrm{co},a})} + \|\dot{\gamma}\|_{g_{\mathrm{co},0}}} \right| ds. \end{aligned} \tag{3.86}$$

We then obtain the estimate

$$\begin{aligned} |\widehat{L}_{\mathrm{co},a}(\widehat{\gamma}) - L_{\mathrm{co},0}(\gamma)| &\leq \int_{\frac{aK}{\rho}}^{1} \left| \frac{\|(\pi^{-1})^*(\widehat{g}_{\mathrm{co},a}) - g_{\mathrm{co},0}\|_{g_{\mathrm{co},0}} \cdot \|\dot{\gamma}\|^2_{g_{\mathrm{co},0}}}{\|\dot{\gamma}\|_{(\pi^{-1})^*(\widehat{g}_{\mathrm{co},a})} + \|\dot{\gamma}\|_{g_{\mathrm{co},0}}} \right| ds \\ &\leq \int_{\frac{aK}{\rho}}^{1} \left| \|(\pi^{-1})^*(\widehat{g}_{\mathrm{co},a}) - g_{\mathrm{co},0}\|_{g_{\mathrm{co},0}} \cdot \|\dot{\gamma}\|_{g_{\mathrm{co},0}} \right| ds. \end{aligned} \tag{3.87}$$

We now use $\|\dot{\gamma}\|_{g_{\mathrm{co},0}} = \rho$, $r\big(\gamma(s)\big) = s \cdot \rho$, and our estimate (3.74) from the Asymptotically Conical Decay Property (CO SR II) to obtain

$$|\widehat{L}_{\mathrm{co},a}(\widehat{\gamma}) - L_{\mathrm{co},0}(\gamma)| \leq \int_{\frac{aK}{\rho}}^{1} \frac{1}{2}\rho \, ds = \frac{1}{2} \cdot (\rho - aK) \tag{3.88}$$

Therefore

$$\widehat{L}_{\mathrm{co},a}(\widehat{\gamma}) \leq |\widehat{L}_{\mathrm{co},a}(\widehat{\gamma}) - L_{\mathrm{co},0}(\gamma)| + L_{\mathrm{co},0}(\gamma) \leq \frac{3}{2} \cdot (\rho - aK). \tag{3.89}$$

Hence, we get

$$\widehat{d}_{\mathrm{co},a}\big(p, \widehat{T}(aK)\big) \leq \frac{3}{2} \cdot (\delta - aK). \tag{3.90}$$

Given two points $p, q \in \widehat{T}(\delta)$, we can connect them using paths that move down along their respective fibers and insert an intermediate path in $\widehat{T}(aK)$. In tandem with our diameter bound (3.78) for $\widehat{T}(aK)$, we get an upper bound

$$\begin{aligned} \widehat{\mathrm{diam}}_{\mathrm{co},a}\big(\widehat{T}(\delta)\big) &\leq a \cdot \widehat{\mathrm{diam}}_{\mathrm{co},1}\big(\widehat{T}(K)\big) + 3 \cdot (\delta - aK) \\ &\leq \Big(\frac{1}{K} \cdot \widehat{\mathrm{diam}}_{\mathrm{co},1}\big(\widehat{T}(K)\big) + 3\Big) \cdot \delta, \end{aligned} \tag{3.91}$$

which is uniformly bounded for $a \in (0, \frac{\delta}{K}]$.

From this, we conclude the following:

**Lemma 3.3.7.** *Fix $K > 0$ such that (3.74) holds. For $\delta > 0$ and $a \in (0, \frac{\delta}{K}]$ there exists a constant $C > 0$ independent of the choice of $\delta$ and $a$ such that*

$$\widehat{\mathrm{diam}}_{\mathrm{co},a}\big(\widehat{T}(\delta)\big) \leq C \cdot \delta. \tag{3.92}$$

### Applying the Main Lemma

The uniform diameter estimate from Lemma 3.3.7 will enable us to prove a useful result akin to that of Song–Weinkove [SW13b].

**Lemma 3.3.8.** *For $0 < \epsilon < 1$, there exists $\delta > 0$ and $a_0 > 0$ such that for $a \in (0, a_0)$, we have*

*i)* $\mathrm{diam}_{\mathrm{co},0}\big(D_0(\delta)\big) < \epsilon$; *and*

*ii)* $\widehat{\mathrm{diam}}_{\mathrm{co},a}\Big(\pi^{-1}\big(D_0(\delta)\big)\Big) < \epsilon$.

*Proof.* We have that $D_0(\delta)$ is a closed disc of radius $\delta$ with respect to a cone metric $g_{\mathrm{co},0} = dr \otimes dr + r^2 \cdot g_L$. Standard arguments from Riemannian geometry show that the diameter of this is is bounded above by $2\delta$. Hence we pick $\delta < \frac{\epsilon}{2}$ to satisfy the first condition.

For the second condition, we first note that $\pi^{-1}\big(D_0(\delta)\big) = \widehat{T}(\delta)$. Lemma 3.3.7 says that there exists a uniform constant $C > 0$ such that

$$\widehat{\mathrm{diam}}_{\mathrm{co},a}\big(\widehat{T}(\delta)\big) \leq C \cdot \delta \tag{3.93}$$

for all $a \in (0, \frac{\delta}{K}]$.

As such, we choose $\delta$ small enough such that $\delta < C^{-1} \cdot \epsilon$, $\delta < \frac{\epsilon}{2}$, and set $a_0 = \frac{\delta}{K}$. The result follows. □

We can now apply Lemma 3.3.7 to prove Gromov–Hausdorff convergence of the three classes of metrics on the small resolution:

i) Convergence of the local models:

$$\big(\widehat{T}(R), \widehat{d}_{\mathrm{co},a}\big) \to \big(D_0(R), d_{\mathrm{co},0}\big)$$

In this case, we only have one ODP singularity $s$. By the diameter estimate for $g_{\mathrm{co},a}$ from Lemma 3.3.8, we see that for each $\epsilon > 0$, we can pick the set $\overline{G} = D_0(\delta)$ for an appropriately small $\delta > 0$ such that Lemma 3.3.5 applies.

ii) Convergence of the Fu–Li–Yau balanced metrics:

$$(\widehat{X}, \widehat{d}_{\mathrm{FLY},a}) \to (X_0, d_{\mathrm{FLY},0})$$

Here we use the fact that the Fu–Li–Yau metrics are, up to scaling, just the Candelas–de la Ossa metrics in a compact set around the $(-1,-1)$-curves $E_i$ and the ODP singularities $s_j$. For $\epsilon > 0$, we can pick $\overline{G_j} = D_0(\delta_j)$ for appropriately small $\delta_j$ around each singular point $s_j$. Coupling this with the smooth convergence of the Fu–Li–Yau metrics on compact sets away from the $(-1,-1)$-curves and singularities, we can apply Lemma 3.3.5.

iii) Convergence of the Hermitian Yang–Mills metrics:

$$(\widehat{X}, \widehat{d}_{\widehat{H}_a}) \to (X_0, d_{H_0})$$

By Proposition 2.1.1, we have the estimate

$$C^{-1} \cdot \widehat{g}_{\mathrm{co},a} \leq \widehat{H}_a \leq C \cdot \widehat{g}_{\mathrm{co},a} \tag{3.94}$$

on the local sets $\widehat{T}(\delta_j)$ around each $(-1,-1)$-curve and $D_0(\delta_j)$ around each singularity $s_j$, where the Fu–Li–Yau metrics are scaled Candelas–de la Ossa metrics. The uniform estimates of Lemma 3.3.8 imply that for $\epsilon > 0$, there exists $\delta_j > 0$ and $a_0 > 0$ such that for all $a \in (0, a_0)$,

$$\mathrm{diam}_{H_0}\big(D_0(\delta_j)\big) < \epsilon \text{ and } \widehat{\mathrm{diam}}_{\widehat{H}_a}\Big(\pi^{-1}\big(D_0(\delta_j)\big)\Big) < \epsilon. \tag{3.95}$$

We can therefore apply Lemma 3.3.5.

### 3.3.4 Estimates on the Smoothings

We now prove the analogous statements on the smoothings $X_t$. Recall that given the conifold transition $\widehat{X} \to X_0 \rightsquigarrow X_t$, we have maps

$$\Phi_t : X_0 \cap \left\{ r(z) > \left(\frac{|t|}{2}\right)^{\frac{1}{3}} \right\} \to X_t \cap \{r(z) > |t|^{\frac{1}{3}}\}. \tag{3.96}$$

In this case, we let the maps $F_t$ be the inverses $\Phi_t^{-1}$, the sets $C_{j,t}$ be the vanishing spheres $D_t(|t|^{\frac{1}{3}})$, and the sets $K_{j,t}$ be the discs $D_0\Big(\big(\frac{|t|}{2}\big)^{\frac{1}{3}}\Big)$.

We once again make use of the local models $(V_t, g_{\mathrm{co},t})$ and $(V_0, g_{\mathrm{co},0})$ to obtain diameter bounds. In particular, we have the "discs"

$$D_t(R) := \{r \leq R\} \subseteq V_t \tag{3.97}$$

and

$$D_0(R) := \{r \leq R\} \subseteq V_0 \tag{3.98}$$

where $r$ denotes the respective radius functions defined in §2. These are related by

$$\Phi_t\big(D_0(R)\big) = D_t(\beta_{t,R}) \tag{3.99}$$

where

$$\beta_{t,R} = \left(R^3 + \frac{|t|^2}{4R^3}\right)^{\frac{1}{3}}. \tag{3.100}$$

As before, we can use the Asmyptotocally Conical Decay Property (CO SM II) to fix a constant $K > 1$ such that

$$\|(\Phi_t)^*(g_{\mathrm{co},t}) - g_{\mathrm{co},0}\|_{g_{\mathrm{co},0}} \leq \frac{1}{2} \tag{3.101}$$

when $r > |t|^{\frac{1}{3}}K$. For $\delta > 0$ and $t$ such that $|t| \in (0, \frac{\delta^3}{K^3})$ it is again useful to split the "disc" $D_t(\beta_{t,\delta})$ as

$$D_t(\beta_{t,\delta}) = D_t\big(\beta_{t,|t|^{\frac{1}{3}}K}\big) \cup \Big(D_t(\beta_{t,\delta}) \setminus D_t\big(\beta_{t,|t|^{\frac{1}{3}}K}\big)\Big). \tag{3.102}$$

Analogously to the case of the small resolution, we will use the Scaling Property (CO SM I) on the "disc" $D_t\big(\beta_{t,|t|^{\frac{1}{3}}K}\big)$, and the Asymptotocally Conical Decay Property (CO SM II) on the "annulus" $D_t(\beta_{t,\delta}) \setminus D_t\big(\beta_{t,|t|^{\frac{1}{3}}K}\big)$.

### Bounds on the Disc

We start with the disc $D_t\big(\beta_{t,|t|^{\frac{1}{3}}K}\big)$. Recall from the Scaling Property (CO SM I) that

$$g_{\mathrm{co},t} = |t|^{\frac{2}{3}} \cdot S^*_{t^{-\frac{1}{3}}}(g_{\mathrm{co},1}). \tag{3.103}$$

As such, we see that

$$S_{t^{-\frac{1}{3}}} : \Big(D_t\big(\beta_{t,|t|^{\frac{1}{3}}K}\big), g_{\mathrm{co},t}\Big) \to \Big(D_1\big(\beta_{1,K}\big), |t|^{\frac{2}{3}} \cdot g_{\mathrm{co},1}\Big) \tag{3.104}$$

is an isometry. From this, it follows that

$$\mathrm{diam}_{\mathrm{co},t}\Big(D_t\big(\beta_{t,|t|^{\frac{1}{3}}K}\big)\Big) = |t|^{\frac{1}{3}} \cdot \mathrm{diam}_{\mathrm{co},1}\big(D_1(\beta_{1,K})\big), \tag{3.105}$$

which is finite since the set $\big(D_1(\beta_{1,K}), g_{\mathrm{co},1}\big)$ is compact and path connected (we can construct a path from any point to the vanishing sphere, which is topologically an $S^3$, and hence path connected).

### "Annular" Bounds

As we did in the case of the small resolution, we will define a convenient class of paths to work with. Let $\widetilde{p} \in D_t(\beta_{t,\delta}) \setminus D_t\big(\beta_{t,|t|^{\frac{1}{3}}K}\big)$ be a point in the "annular" region. We construct a curve $\widetilde{\gamma}$ from $D_t\big(\beta_{t,|t|^{\frac{1}{3}}K}\big)$ to $\widetilde{p}$ and estimate its length $L_{\mathrm{co},t}(\widetilde{\gamma})$ with respect to $g_{\mathrm{co},t}$. To do this, we pullback to the cone $V_0$ and use a radial ray.

Since $\Phi_t$ is a diffeomorphism on the "annular" region, we can write $\widetilde{p} = \Phi_t(p)$ for some $p \in V_0$. Let $\rho = r(p)$. We note that our choice of point $\widetilde{p}$ is such that $\rho \in (|t|^{\frac{1}{3}}K, \delta]$.

We define a path $\gamma : [\frac{|t|^{\frac{1}{3}}K}{\rho}, 1] \to V_0$ by

$$\gamma(s) = s^{\frac{3}{2}} \cdot p \tag{3.106}$$

and so

$$r\big(\gamma(s)\big) = s \cdot \rho. \tag{3.107}$$

This path is chosen to mirror the one defined for the small resolution. In particular, it begins in $D_0(|t|^{\frac{1}{3}}K)$ and moves outward along a ray until it reaches $\gamma(1) = p$.

The analogous results of Lemma 3.3.6 hold for this curve, and so we have

$$\|\dot{\gamma}\|_{g_{\mathrm{co},0}} = \rho \text{ and } L_{\mathrm{co},0}(\gamma) = \rho - |t|^{\frac{1}{3}}K. \tag{3.108}$$

We can pull this curve back to $V_t$ and define $\widetilde{\gamma} = \Phi_t \circ \gamma$. As we did for the small resolution, we compare the lengths of the curve $\gamma$ and its pullback $\widetilde{\gamma}$ with respect to their corresponding metrics $g_{\mathrm{co},0}$ and $g_{\mathrm{co},t}$.

Similar computations to §3.3.3 show that

$$|L_{\mathrm{co},t}(\widetilde{\gamma}) - L_{\mathrm{co},0}(\gamma)| \leq \int_{\frac{|t|^{\frac{1}{3}}K}{\rho}}^{1} \|(\Phi_t)^*(g_{\mathrm{co},t}) - g_{\mathrm{co},0}\|_{g_{\mathrm{co},0}} \cdot \|\dot{\gamma}\|_{g_{\mathrm{co},0}}\, ds. \tag{3.109}$$

Applying what we know about the curve $\gamma$ and the estimate (3.101) we get

$$|L_{\mathrm{co},t}(\widetilde{\gamma}) - L_{\mathrm{co},0}(\gamma)| \leq \int_{\frac{|t|^{\frac{1}{3}}K}{\rho}}^{1} \frac{1}{2}\rho\, ds = \frac{1}{2} \cdot (\rho - |t|^{\frac{1}{3}}K). \tag{3.110}$$

As such, we see that

$$L_{co,t}(\widetilde{\gamma}) \leq |L_{\mathrm{co},t}(\widetilde{\gamma}) - L_{\mathrm{co},0}(\gamma)| + L_{\mathrm{co},0}(\gamma) \leq \frac{3}{2} \cdot (\rho - |t|^{\frac{1}{3}}K) \tag{3.111}$$

and so

$$d_{\mathrm{co},t}\Big(\widetilde{p}, D_t\big(\beta_{t,|t|^{\frac{1}{3}}K}\big)\Big) \leq \frac{3}{2} \cdot (\delta - |t|^{\frac{1}{3}}K). \tag{3.112}$$

We can use two of these curves concatenated with an intermediate path in $D_t\big(\beta_{t,|t|^{\frac{1}{3}}K}\big)$ to connect any two points $\widetilde{p}, \widetilde{q} \in D_t(\beta_{t,\delta})$. Hence combining the above bound with the diameter bound (3.78) yields

$$\begin{aligned} \mathrm{diam}_{\mathrm{co},t}\big(D_t(\beta_{t,\delta})\big) &\leq |t|^{\frac{1}{3}} \cdot \mathrm{diam}_{\mathrm{co},1}\big(D_1(\beta_{1,K})\big) + 3 \cdot (\delta - |t|^{\frac{1}{3}}K) \\ &\leq \Big(\frac{1}{K} \cdot \mathrm{diam}_{\mathrm{co},1}\big(D_1(\beta_{1,K})\big) + 3\Big) \cdot \delta. \end{aligned} \tag{3.113}$$

We get the following result:

**Lemma 3.3.9.** *Fix $K > 0$ such that* (3.101) *holds. For $\delta > 0$ and $t$ with $|t| \in (0, \frac{\delta^3}{K^3}]$ there exists a constant $C > 0$ independent of the choice of $\delta$ and $t$ such that*

$$\operatorname{diam}_{\mathrm{co},t}\big(D_t(\beta_{t,\delta})\big) \leq C \cdot \delta. \tag{3.114}$$

### Bounds fo the Fu–Li–Yau Metrics

On the smoothings $X_t$, the Fu–Li–Yau metrics $g_{\mathrm{FLY},t}$ are only close to scaled Candelas–de la Ossa metrics $g_{\mathrm{co},t}$, instead of being exactly equal to them. Because of this, we require a version of the diameter bound (3.114) for the Fu–Li–Yau metrics. This will follow by virtue of the estimate

$$\sup_{\{r \leq R_0\}} \|g_{\mathrm{FLY},t} - c_j \cdot g_{\mathrm{co},t}\|_{g_{\mathrm{co},t}} \leq C \cdot |t|^{\frac{2}{3}} \tag{3.115}$$

near each vanishing sphere from the Local Model Property (FLY SM I) .

Consider a curve $\gamma : [0,1] \to D_t(\beta_{t,\delta})$. We compare the length of this path $\gamma$ with respect to $g_{\mathrm{FLY},t}$ and $c \cdot g_{\mathrm{co},t}$.

$$|L_{\mathrm{FLY},t}(\gamma) - \sqrt{c_j} \cdot L_{\mathrm{co},t}(\gamma)| \leq \frac{1}{\sqrt{c_j}} \cdot \int_0^1 \|g_{\mathrm{FLY},t} - c_j \cdot g_{\mathrm{co},t}\|_{g_{\mathrm{co},t}} \cdot \|\dot{\gamma}\|_{g_{\mathrm{co},t}}\, ds. \tag{3.116}$$

We have a finite number of vanishing spheres and so the values of $c_j$ are bounded. Using the estimate, and ensuring $t$ and $\delta$ are such that $\beta_{t,\delta} < R_0$ we see that

$$|L_{\mathrm{FLY},t}(\gamma) - \sqrt{c} \cdot L_{\mathrm{co},t}(\gamma)| \leq C \cdot |t|^{\frac{2}{3}} L_{\mathrm{co},t}(\gamma). \tag{3.117}$$

Hence

$$L_{\mathrm{FLY},t}(\gamma) \leq C \cdot (|t|^{\frac{2}{3}} + 1) \cdot L_{\mathrm{co},t}(\gamma), \tag{3.118}$$

and so

$$\operatorname{diam}_{\mathrm{FLY},t}\big(D_t(\beta_{t,\delta})\big) \leq C \cdot (|t|^{\frac{2}{3}} + 1) \cdot \operatorname{diam}_{\mathrm{co},t}\big(D_t(\beta_{t,\delta})\big). \tag{3.119}$$

**Lemma 3.3.10.** *Fix $K > 0$ such that* (3.101) *holds. For sufficiently small $\delta > 0$ and $t$ with $|t| \in (0, \frac{\delta^3}{K^3}]$ there exists a constant $C > 0$ independent of the choice of $\delta$ and $t$ such that*

$$\operatorname{diam}_{\mathrm{FLY},t}\big(D_t(\beta_{t,\delta})\big) \leq C \cdot \delta \cdot (\delta^2 + 1). \tag{3.120}$$

### Applying the Main Lemma

We now prove an analogue of Lemma 3.3.8 in the case of the of the smoothings for both the Candelas–de la Ossa metrics $g_{\mathrm{co},t}$ and the Fu–Li–Yau metrics $g_{\mathrm{FLY},t}$.

**Lemma 3.3.11.** *For $0 < \epsilon < 1$, there exists $\delta > 0$ and $t_0 > 0$ such that for $t$ with $|t| \in (0, t_0)$, we have*

*i)* $\mathrm{diam}_{\mathrm{co},0}\big(D_0(\delta)\big) < \epsilon$*; and*

*ii)* $\mathrm{diam}_{\mathrm{co},t}\big(D_t(\beta_{t,\delta})\big) < \epsilon$.

*The result also holds when using the Fu–Li–Yau metrics $g_{\mathrm{FLY},0}$ and $g_{\mathrm{FLY},t}$ instead of the Candelas–de la Ossa metrics $g_{\mathrm{co},0}$ and $g_{\mathrm{co},t}$.*

*Proof.* As was the case for Lemma 3.3.8, the first condition holds as long as $\delta < \frac{\epsilon}{2}$. Using either (3.114) or (3.120), we can see that the result holds by setting $t_0 = \frac{\delta^3}{K^3}$ as long as $\delta$ is sufficiently small. □

With the above lemma, we can show the Gromov–Hausdorff convergence of the metrics on the smoothings by appealing to the Main Lemma (Lemma 3.3.5):

i) Convergence of the local models:

$$\big(D_t(\beta_{t,R}), d_{\mathrm{co},t}\big) \to \big(D_0(R), d_{\mathrm{co},0}\big)$$

We only have one ODP singularity $s$ for this case. The diameter estimate for $g_{\mathrm{co},t}$ from Lemma 3.3.11 tells us that for each $\epsilon > 0$, we can pick the set $\overline{G} = D_0(\delta)$ for an appropriately small $\delta > 0$ such that Lemma 3.3.5 applies.

ii) Convergence of the Fu–Li–Yau balanced metrics:

$$(\widehat{X}, d_{\mathrm{FLY},t}) \to (X_0, d_{\mathrm{FLY},0})$$

Again, using the estimate from Lemma 3.3.11, we see that for $\epsilon > 0$, we can pick $\overline{G_j} = D_0(\delta_j)$ for appropriately small $\delta_j$ around each singular point $s_j$. Coupling this with the smooth convergence of the Fu–Li–Yau metrics on compact sets away from the vanishing spheres and singularities, we can apply Lemma 3.3.5.

iii) Convergence of the Hermitian Yang–Mills metrics:

$$(\widehat{X}, \widehat{d}_{\widehat{H}_a}) \to (X_0, d_{H_0})$$

Recall the uniform equivalence of metrics

$$C^{-1} \cdot g_{\mathrm{FLY},t} \leq H_t \leq C \cdot g_{\mathrm{FLY},t} \tag{3.121}$$

from §2.2.3. The uniform estimates of Lemma 3.3.11 then tell us that for $\epsilon > 0$, there exists $\delta_j > 0$ for each vanishing sphere and $t_0 > 0$ such that when $|t| \in (0, t_0)$,

$$\mathrm{diam}_{H_0}\big(D_0(\delta_j)\big) < \epsilon \text{ and } \mathrm{diam}_{H_t}\Big(D_t(\beta_{t,\delta_j})\big)\Big) < \epsilon. \tag{3.122}$$

Applying the Main Lemma (Lemma 3.3.5) yields the result.

By combining the results above results and those at the end of §3.3.3, we obtain Theorem 3.3.1.

# Chapter 4

# The Anomaly Flow

As previously noted, the metrics of Fu–Li–Yau [FLY12] and Collins–Picard–Yau [CPY24] only partially solve the Hull–Strominger system (1.22) - (1.25). The missing condition in these constructions was the heterotic Bianchi identity

$$\sqrt{-1}\partial\overline{\partial}\omega - \alpha'\Big(\mathrm{tr}(\mathrm{Rm}\wedge\mathrm{Rm}) - \mathrm{tr}(F\wedge F)\Big) = 0. \tag{4.1}$$

In this chapter, we discuss an approach to finding solutions to this equation by geometric flows. In particular, we will show a condition on the slope parameter $\alpha'$ that will ensure that the flow can be extended.

## 4.1 Evolution Equations

In an effort to find solutions to the Hull–Strominger system, Phong–Picard–Zhang [PPZ18c] have proposed a geometric flows approach using the so-called Anomaly flow. This is a geometric flow on a Calabi–Yau threefold $X$ with nowhere-vanishing $(3,0)$-form $\Upsilon$ that evolves a Hermitian metric $\omega$ by

$$\frac{\partial}{\partial t}\big(\|\Upsilon\|_\omega \omega^2\big) = \sqrt{-1}\partial\overline{\partial}\omega - \alpha'\big(\mathrm{tr}(\mathrm{Rm}\wedge\mathrm{Rm}) - \Phi\big). \tag{4.2}$$

Here $\Phi$ is a prescibed $(2,2)$-form in $c_2(X)$ that may evolve with time. By Chern–Weil Theory, the RHS of the evolution equation is closed and thus the Anomaly flow preserves the conformally balanced condition.

**Remark 4.1.1.** In order to solve the full system, we can couple the Anomaly flow with another flow of Hermitian metrics $H$ on a holomorphic bundle $E \to X$. In particular, as proposed in [PPZ18c], we may set

$$H^{-1}\frac{\partial}{\partial t}H = -\Lambda_\omega F, \tag{4.3}$$

and set $\Phi = \mathrm{tr}(F\wedge F)$ in the Anomaly flow. Given appropriate initial conditions, the stationary points of the coupled flow can be checked to satisfy the Hull–Strominger system for the slope parameter $\alpha'$.

Given a geometric flow, we have the usual questions about its short-time existence and uniqueness, long-time existence, and convergence of solutions. The short-time existence and uniqueness of the Anomaly flow has been shown in [PPZ18c], while long-time existence has been shown in various settings (see [PPZ18a, PPZ18b, PPZ19b, FHP21] and also [Puj21, PU21] for versions using non-Chern connections). The rest of this chapter will focus on a general long-time existence result for the flow.

In [PPZ18b], Phong–Picard–Zhang rewrite the evolution equation (4.2) in terms of the metric $g$.

**Theorem 4.1.2** (Phong–Picard–Zhang [PPZ18b])**.** *Under the Anomaly flow* (4.2)*, the metric $g$ evolves by*

$$\frac{\partial}{\partial t} g_{p\bar{q}} = \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot \Big[ - \widetilde{R}_{p\bar{q}} + g^{\alpha\bar{\beta}} g^{r\bar{s}} T_{pr\bar{\beta}} \overline{T}_{\alpha\bar{q}\bar{s}} - \alpha' g^{r\bar{s}} \big( R_{[r\bar{q}}{}^{\alpha}{}_{\beta} R_{p\bar{s}]}{}^{\beta}{}_{\alpha} - \Phi_{r\bar{q}p\bar{s}} \big) \Big]. \tag{4.4}$$

*Here $R_{p\bar{q}}{}^{r}{}_{s} = -\partial_{\bar{q}}(g^{r\bar{m}} \partial_p g_{s\bar{m}})$ is the Chern curvature tensor, $T_{kp\bar{q}} = \partial_k g_{p\bar{q}} - \partial_p g_{k\bar{q}}$ is the torsion tensor, and $\widetilde{R}_{p\bar{q}} = R_{p\bar{q}}{}^{k}{}_{k}$ is (one notion of) the Ricci curvature.*

This can schematically be written as

$$\frac{\partial}{\partial t} g = \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot \Big[\mathrm{Rm} + T * \overline{T} + \alpha' \cdot \big(\mathrm{Rm} * \mathrm{Rm} + \Phi\big)\Big], \tag{4.5}$$

where $*$ denotes a finite linear combination of contractions using the metric $g$.

The results of Theorem 4 and 5 of [PPZ18b] respectively yield evolution equations for the curvature and torsion along the Anomaly flow:

$$\frac{\partial}{\partial t} \mathrm{Rm} = \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot \Big[\frac{1}{2}\Delta_R \mathrm{Rm} + H_1 + \alpha' \cdot \big(\nabla\overline{\nabla}(\mathrm{Rm} * \mathrm{Rm}) + H_2\big)\Big], \tag{4.6}$$

$$\frac{\partial}{\partial t} T = \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot \Big[\frac{1}{2}\Delta_R T + K_1 + \alpha' \cdot \big(\nabla(\mathrm{Rm} * \mathrm{Rm}) + K_2\big)\Big]. \tag{4.7}$$

Here we have

$$\begin{aligned} H_1 &= \nabla\overline{\nabla}(T * \overline{T}) + \overline{\nabla}(T * \mathrm{Rm}) + \nabla(\overline{T} * \mathrm{Rm}) \\ &\quad + \mathrm{Rm} * \mathrm{Rm} + \overline{\nabla}(T * T * \overline{T}) + \nabla(\overline{T} * \overline{T} * T) \\ &\quad + T * \overline{T} * \mathrm{Rm} + T * \overline{T} * T * \overline{T}, \end{aligned} \tag{4.8}$$

$$\begin{aligned} H_2 &= \nabla\overline{\nabla}\Phi + \mathrm{Rm} * \Phi + \overline{\nabla}(T * \mathrm{Rm} * \mathrm{Rm}) + \nabla(\overline{T} * \mathrm{Rm} * \mathrm{Rm}) \\ &\quad + \mathrm{Rm} * \mathrm{Rm} * \mathrm{Rm} + \overline{\nabla}(T * \Phi) + \nabla(\overline{T} * \Phi) \\ &\quad + T * \overline{T} * \mathrm{Rm} * \mathrm{Rm} + T * \overline{T} * \Phi, \end{aligned} \tag{4.9}$$

which involve at most 2 covariant derivatives of $T$ and at most 1 covariant derivative of Rm.

Similarly, we have

$$K_1 = \nabla(T * \overline{T}) + T * \mathrm{Rm} + T * T * \overline{T}, \tag{4.10}$$

$$K_2 = \nabla\Phi + T * \mathrm{Rm} * \mathrm{Rm} + T * \Phi, \tag{4.11}$$

which involve at most 1 covariant derivative of $T$ and none of Rm.

By the evolution of the Chern connection, we also have expressions for the evolution of covariant derivatives of both Rm and $T$.

$$\begin{aligned} &\frac{\partial}{\partial t}(\nabla^m\overline{\nabla}^l \mathrm{Rm}) \\ &= \sum_{i+j>0}\sum_{i=0}^{m}\sum_{j=0}^{l}(\nabla^{m-i}\overline{\nabla}^{l-j}\mathrm{Rm}) * \left(\nabla^i\overline{\nabla}^j\left(\frac{\partial}{\partial t}g\right)\right) \\ &\quad + \nabla^m\overline{\nabla}^l\left(\left(\frac{1}{2\|\Upsilon\|_\omega}\right)\cdot\Big[\frac{1}{2}\Delta_R\mathrm{Rm} + H_1\right. \\ &\qquad\qquad \left. + \alpha'\cdot\left(\nabla\overline{\nabla}(\mathrm{Rm} * \mathrm{Rm}) + H_2\right)\Big]\right), \end{aligned} \tag{4.12}$$

$$\begin{aligned}
&\frac{\partial}{\partial t}(\nabla^m\overline{\nabla}^l T)\\
&= \sum_{i+j>0}\sum_{i=0}^{m}\sum_{j=0}^{l}(\nabla^{m-i}\overline{\nabla}^{l-j}T) * \Big(\nabla^i\overline{\nabla}^j\Big(\frac{\partial}{\partial t}g\Big)\Big)\\
&\quad + \nabla^m\overline{\nabla}^l\Bigg(\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot\Big[\frac{1}{2}\Delta_R T + K_1\\
&\qquad\qquad + \alpha'\cdot\big(\nabla(\mathrm{Rm} * \mathrm{Rm}) + K_2\big)\Big]\Bigg). \qquad (4.13)
\end{aligned}$$

## 4.2 Integral Shi-Type Estimates

In this section, we work to obtain uniform $L^\infty$-estimates for covariant derivatives of Rm and $T$ along the Anomaly flow. To do this, we first assume some base level of regularity along the flow.

### 4.2.1 Base Assumptions and Notation

Suppose for $k \geq 1$ that these exist positive constants $B, C_0, C_1, \ldots, C_{k-1}$ such that

$$B^{-1} \leq \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \leq B, \qquad (4.14)$$

$$\|T\|, \|\overline{T}\|, \|\mathrm{Rm}\|, \|DT\|, \|D\overline{T}\| \leq C_0, \qquad (4.15)$$

$$\|D^q\mathrm{Rm}\|, \|D^{q+1}T\|, \|D^{q+1}\overline{T}\| \leq C_q \text{ for } 1 \leq q \leq k-1 \qquad (4.16)$$

along the Anomaly flow on the interval $[0, \tau)$. Here all norms are taken with respect to the evolving metric and $D^qA$ denotes all combinations of $q$th-order covariant derivatives of a tensor $A$ such that:

$$\|D^qA\|^2 = \sum_{m+l=q}\|\nabla^m\overline{\nabla}^l A\|^2. \qquad (4.17)$$

**Remark 4.2.1.** We note that the first assumption (4.14) and the second assumption (4.15), in conjunction with (4.4) imply that $\big\|\frac{\partial}{\partial t}g\big\|$ is uniformly bounded along the flow. It follows that the evolving metric $g$ is uniformly bounded above and below by the initial metric $g_0$. As such, the volume $\int_X 1\, d\mathrm{vol}$ is also uniformly bounded along the flow.

In the following computations, we assume that the evolving form $\Phi$ satisfies any required regularity conditions for simplicity.

For the rest of this chapter, we adopt the convention that $C$ denotes a generic positive constant that may change from line to line and may depend on $\alpha'$ but does not depend on the time $t$. Furthermore, all integrals will be taken with respect to the evolving volume form and as such, we omit the volume form to reduce clutter.

### 4.2.2 Estimates on $\|D^k \mathrm{Rm}\|$ and $\|D^{k+1}T\|$

Given the assumptions (4.14) - (4.16), we aim to prove the existence of a positive constant $C_k$ such that

$$\|D^k \mathrm{Rm}\|, \|D^{k+1}T\|, \|D^{k+1}\overline{T}\| \leq C_k. \tag{4.18}$$

We will do this by first obtaining $L^{2p}$-bounds under appropriate conditions on the slope parameter $\alpha'$. To upgrade the $L^{2p}$-bounds to full-fledged $L^\infty$-bounds, we will also require $L^{2p}$-bounds on higher-order derivatives of Rm and $T$. These will be obtained in the following section (§4.2.3).

From the expression (A.16) and the regularity of $T$, we see that

$$\left\| D^q \Big( \frac{1}{2\|\Upsilon\|_\omega} \Big) \right\| \leq C \text{ for } 0 \leq q \leq k+1. \tag{4.19}$$

The commutator identity (A.18) also gives that

$$\|D^{k+1}\overline{T}\| \leq C + \|D^{k+1}T\|, \tag{4.20}$$

$$\|D^{k+2}\overline{T}\| \leq C + C \cdot \|D^k \mathrm{Rm}\| + \|D^{k+2}T\|. \tag{4.21}$$

From this, we only need to worry about bounding Rm and $T$ since inequalities for $\overline{T}$ will follow.

#### Pointwise Estimates

We begin by working with pointwise bounds as in [PPZ18b, PPZ18c]. The main difference is that in our setting, the terms involving the slope parameter $\alpha'$ cannot be as easily dealt with as we shall see from the upcoming computations. Despite this issue, these extra terms can be written as divergences, and we shall integrate by parts to deal with them.

In the ensuing section, we suppose that $m+l = k \geq 2$. This is to ensure that the terms that appear are no more than quadratic in unknown quantities that we want to bound. We will tackle the case where $k = 1$ later in §4.3.2.

Using (4.6) and (4.12) and later applying the CBS Inequality, we can check that

$$
\begin{aligned}
&\frac{\partial}{\partial t}\|\nabla^m\overline{\nabla}^l\mathrm{Rm}\|^2 \\
&= C\cdot\Big\|\frac{\partial}{\partial t}g\Big\|\cdot\|\nabla^m\overline{\nabla}^l\mathrm{Rm}\|^2 + 2\mathrm{Re}\Big\langle\frac{\partial}{\partial t}(\nabla^m\overline{\nabla}^l\mathrm{Rm}),\nabla^m\overline{\nabla}^l\mathrm{Rm}\Big\rangle \\
&\leq C\cdot\Big\|\frac{\partial}{\partial t}g\Big\|\cdot\|\nabla^m\overline{\nabla}^l\mathrm{Rm}\|^2 \\
&\quad+\sum_{i+j>0}\sum_{i=0}^{m}\sum_{j=0}^{l}2\mathrm{Re}\Big\langle(\nabla^{m-i}\overline{\nabla}^{l-j}\mathrm{Rm})*\Big(\nabla^i\overline{\nabla}^j\Big(\frac{\partial}{\partial t}g\Big)\Big),\nabla^m\overline{\nabla}^l\mathrm{Rm}\Big\rangle \\
&\quad+2\mathrm{Re}\Big\langle\nabla^m\overline{\nabla}^l\Big(\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot H_1\Big),\nabla^m\overline{\nabla}^l\mathrm{Rm}\Big\rangle \\
&\quad+2\alpha'\mathrm{Re}\Big\langle\nabla^m\overline{\nabla}^l\Big(\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot H_2\Big),\nabla^m\overline{\nabla}^l\mathrm{Rm}\Big\rangle \\
&\quad+\sum_{i+j<k}\sum_{i=0}^{m}\sum_{j=0}^{l}2\mathrm{Re}\Big\langle\Big(\nabla^{m-i}\overline{\nabla}^{l-j}\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\Big) \\
&\qquad\qquad\qquad *\Big(\nabla^i\overline{\nabla}^j\Big(\frac{1}{2}\Delta_R\mathrm{Rm}\Big)\Big),\nabla^m\overline{\nabla}^l\mathrm{Rm}\Big\rangle \\
&\quad+\sum_{i+j<k}\sum_{i=0}^{m}\sum_{j=0}^{l}2\alpha'\mathrm{Re}\Big\langle\Big(\nabla^{m-i}\overline{\nabla}^{l-j}\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\Big) \\
&\qquad\qquad\qquad *\Big(\nabla^i\overline{\nabla}^j(\nabla\overline{\nabla}(\mathrm{Rm}*\mathrm{Rm}))\Big),\nabla^m\overline{\nabla}^l\mathrm{Rm}\Big\rangle \\
&\quad+\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot 2\mathrm{Re}\Big\langle\nabla^m\overline{\nabla}^l\Big(\frac{1}{2}\Delta_R\mathrm{Rm}\Big),\nabla^m\overline{\nabla}^l\mathrm{Rm}\Big\rangle \\
&\quad+\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot 2\alpha'\mathrm{Re}\Big\langle\nabla^m\overline{\nabla}^l(\nabla\overline{\nabla}(\mathrm{Rm}*\mathrm{Rm})),\nabla^m\overline{\nabla}^l\mathrm{Rm}\Big\rangle \\
&\leq \underbrace{C\cdot\Big\|\frac{\partial}{\partial t}g\Big\|\cdot\|\nabla^m\overline{\nabla}^l\mathrm{Rm}\|^2}_{\textbf{(I)}} \\
&\quad+\underbrace{\sum_{i+j>0}\sum_{i=0}^{m}\sum_{j=0}^{l}C\cdot\Big\|\nabla^i\overline{\nabla}^j\Big(\frac{\partial}{\partial t}g\Big)\Big\|\cdot\|\nabla^m\overline{\nabla}^l\mathrm{Rm}\|}_{\textbf{(II)}} \\
&\quad+\underbrace{C\cdot\Big\|\nabla^m\overline{\nabla}^l\Big(\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot H_1\Big)\Big\|\cdot\|\nabla^m\overline{\nabla}^l\mathrm{Rm}\|}_{\textbf{(III)}}
\end{aligned}
$$

$$+ \underbrace{C \cdot \Big\| \nabla^m \overline{\nabla}^l \Big( \Big( \frac{1}{2\|\Upsilon\|_\omega} \Big) \cdot H_2 \Big) \Big\| \cdot \| \nabla^m \overline{\nabla}^l \mathrm{Rm} \|}_{(\mathbf{IV})}$$

$$+ \underbrace{\sum_{i+j<k} \sum_{i=0}^{m} \sum_{j=0}^{l} C \cdot \| \nabla^i \overline{\nabla}^j (\Delta_R \mathrm{Rm}) \| \cdot \| \nabla^m \overline{\nabla}^l \mathrm{Rm} \|}_{(\mathbf{V})}$$

$$+ \underbrace{\sum_{i+j<k} \sum_{i=0}^{m} \sum_{j=0}^{l} C \cdot \| \nabla^i \overline{\nabla}^j (\nabla \overline{\nabla} \mathrm{Rm}) \| \cdot \| \nabla^m \overline{\nabla}^l \mathrm{Rm} \|}_{(\mathbf{VI})}$$

$$+ \underbrace{\Big( \frac{1}{2\|\Upsilon\|_\omega} \Big) \cdot 2\mathrm{Re} \Big\langle \nabla^m \overline{\nabla}^l \Big( \frac{1}{2} \Delta_R \mathrm{Rm} \Big), \nabla^m \overline{\nabla}^l \mathrm{Rm} \Big\rangle}_{(\mathbf{VII})}$$

$$+ \underbrace{\Big( \frac{1}{2\|\Upsilon\|_\omega} \Big) \cdot 2\alpha' \mathrm{Re} \Big\langle \nabla^m \overline{\nabla}^l \big( \nabla \overline{\nabla} (\mathrm{Rm} * \mathrm{Rm}) \big), \nabla^m \overline{\nabla}^l \mathrm{Rm} \Big\rangle}_{(\mathbf{VIII})}. \tag{4.22}$$

We work to bound each of the eight terms.

**Terms (I) - (VI)** The terms here are relatively well-behaved. In short, this is because these all have an appropriate amount of covariant derivatives that can be handled. We recall that

$$\frac{\partial}{\partial t} g = \Big( \frac{1}{2\|\Upsilon\|_\omega} \Big) \cdot \Big[ \mathrm{Rm} + T * \overline{T} + \alpha' \cdot \big( \mathrm{Rm} * \mathrm{Rm} + \Phi \big) \Big], \tag{4.23}$$

$$\begin{aligned} H_1 &= \nabla \overline{\nabla} (T * \overline{T}) + \overline{\nabla} (T * \mathrm{Rm}) + \nabla (\overline{T} * \mathrm{Rm}) \\ &\quad + \mathrm{Rm} * \mathrm{Rm} + \overline{\nabla} (T * T * \overline{T}) + \nabla (\overline{T} * \overline{T} * T) \\ &\quad + T * \overline{T} * \mathrm{Rm} + T * \overline{T} * T * \overline{T}, \end{aligned} \tag{4.24}$$

$$\begin{aligned} H_2 &= \nabla \overline{\nabla} \Phi + \mathrm{Rm} * \Phi + \overline{\nabla} (T * \mathrm{Rm} * \mathrm{Rm}) + \nabla (\overline{T} * \mathrm{Rm} * \mathrm{Rm}) \\ &\quad + \mathrm{Rm} * \mathrm{Rm} * \mathrm{Rm} + \overline{\nabla} (T * \Phi) + \nabla (\overline{T} * \Phi) \\ &\quad + T * \overline{T} * \mathrm{Rm} * \mathrm{Rm} + T * \overline{T} * \Phi. \end{aligned} \tag{4.25}$$

Using the above and our initial assumptions from §4.2.1, we see that

$$\Big\| \nabla^i \overline{\nabla}^j \Big( \frac{\partial}{\partial t} g \Big) \Big\| \leq \begin{cases} C, & \text{if } i + j \leq k - 1; \text{ and} \\ C + C \cdot \| D^k \mathrm{Rm} \|, & \text{if } i + j = k. \end{cases} \tag{4.26}$$

As such, we get that

$$(\mathbf{I}) \leq C \cdot \|D^k\mathrm{Rm}\|^2, \tag{4.27}$$
$$(\mathbf{II}) \leq C \cdot \|D^k\mathrm{Rm}\| + C \cdot \|D^k\mathrm{Rm}\|^2. \tag{4.28}$$

As before, we note that $H_1$ and $H_2$ involve at most 2 covariant derivatives of $T$ and 1 of Rm. From this, we see that

$$\begin{aligned}\Big\|\nabla^m\overline{\nabla}^l\Big(\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot H_1\Big)\Big\| &\leq C + C\cdot\|D^k\mathrm{Rm}| + C\cdot\|D^{k+1}T\| \\ &\quad + C\cdot\|D^{k+1}\mathrm{Rm}\| + C\cdot\|D^{k+2}T\|, \end{aligned}\tag{4.29}$$

$$\begin{aligned}\Big\|\nabla^m\overline{\nabla}^l\Big(\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot H_2\Big)\Big\| &\leq C + C\cdot\|D^k\mathrm{Rm}| + C\cdot\|D^{k+1}T\| \\ &\quad + C\cdot\|D^{k+1}\mathrm{Rm}\|. \end{aligned}\tag{4.30}$$

This gives that

$$\begin{aligned}(\mathbf{III}) &\leq C\cdot\|D^k\mathrm{Rm}\| + C\cdot\|D^k\mathrm{Rm}|^2 + C\cdot\|D^k\mathrm{Rm}\|\cdot\|D^{k+1}T\| \\ &\quad + C\cdot\|D^k\mathrm{Rm}\|\cdot\|D^{k+1}\mathrm{Rm}\| + C\cdot\|D^k\mathrm{Rm}\|\cdot\|D^{k+2}T\|, \end{aligned}\tag{4.31}$$

$$\begin{aligned}(\mathbf{IV}) &\leq C\cdot\|D^k\mathrm{Rm}\| + C\cdot\|D^k\mathrm{Rm}\|^2 + C\cdot\|D^k\mathrm{Rm}\|\cdot\|D^{k+1}T\| \\ &\quad + C\cdot\|D^k\mathrm{Rm}\|\cdot\|D^{k+1}\mathrm{Rm}\|. \end{aligned}\tag{4.32}$$

A similar analysis tells us that

$$(\mathbf{V}) \leq C\cdot\|D^k\mathrm{Rm}\| + C\cdot\|D^k\mathrm{Rm}|^2 + C\cdot\|D^k\mathrm{Rm}\|\cdot\|D^{k+1}\mathrm{Rm}\|, \tag{4.33}$$
$$(\mathbf{VI}) \leq C\cdot\|D^k\mathrm{Rm}\| + C\cdot\|D^k\mathrm{Rm}\|^2 + C\cdot\|D^k\mathrm{Rm}\|\cdot\|D^{k+1}\mathrm{Rm}\|. \tag{4.34}$$

**Term (VII)** A general method for dealing with this term is to isolate the highest-order parts while extracting a Laplacian and good negative terms.

Applying the Commutator Identity (A.17), we have

$$\begin{aligned}\nabla^m\overline{\nabla}^l(\Delta_R\mathrm{Rm}) &= \Delta_R(\nabla^m\overline{\nabla}^l\mathrm{Rm}) + \sum_{i=0}^m\sum_{j=0}^l(\nabla^{m-i}\overline{\nabla}^{l-j}\mathrm{Rm}) * (\nabla^i\overline{\nabla}^j\mathrm{Rm}) \\ &\quad + \sum_{i=0}^m\sum_{j=0}^l(\nabla^{m-i}\overline{\nabla}^{l+1-j}\mathrm{Rm}) * (\nabla^i\overline{\nabla}^j T) \\ &\quad + \sum_{i=0}^m\sum_{j=0}^l(\nabla^{m+1-i}\overline{\nabla}^{l-j}\mathrm{Rm}) * (\nabla^i\overline{\nabla}^j \overline{T}). \end{aligned}\tag{4.35}$$

This gives

$$
\begin{aligned}
&\left(\frac{1}{2\|\Upsilon\|_\omega}\right) \cdot 2\mathrm{Re}\left\langle \nabla^m \overline{\nabla}^l \left(\frac{1}{2}\Delta_R \mathrm{Rm}\right), \nabla^m \overline{\nabla}^l \mathrm{Rm}\right\rangle \\
&\leq \left(\frac{1}{2\|\Upsilon\|_\omega}\right) \cdot \mathrm{Re}\left\langle \Delta_R(\nabla^m \overline{\nabla}^l \mathrm{Rm}), \nabla^m \overline{\nabla}^l \mathrm{Rm}\right\rangle \\
&\quad + \sum_{i=0}^{m} \sum_{j=0}^{l} C \cdot \|\nabla^{m-i} \overline{\nabla}^{l-j} \mathrm{Rm}\| \cdot \|\nabla^i \overline{\nabla}^j \mathrm{Rm}\| \cdot \|\nabla^m \overline{\nabla}^l \mathrm{Rm}\| \\
&\quad + \sum_{i=0}^{m} \sum_{j=0}^{l} C \cdot \|\nabla^{m-i} \overline{\nabla}^{l+1-j} \mathrm{Rm}\| \cdot \|\nabla^i \overline{\nabla}^j T\| \cdot \|\nabla^m \overline{\nabla}^l \mathrm{Rm}\| \\
&\quad + \sum_{i=0}^{m} \sum_{j=0}^{l} C \cdot \|\nabla^{m+1-i} \overline{\nabla}^{l-j} \mathrm{Rm}\| \cdot \|\nabla^i \overline{\nabla}^j \overline{T}\| \cdot \|\nabla^m \overline{\nabla}^l \mathrm{Rm}\| \\
&\leq \left(\frac{1}{2\|\Upsilon\|_\omega}\right) \cdot \mathrm{Re}\left\langle \Delta_R(\nabla^m \overline{\nabla}^l \mathrm{Rm}), \nabla^m \overline{\nabla}^l \mathrm{Rm}\right\rangle \\
&\quad + C \cdot \|D^k \mathrm{Rm}\| + C \cdot \|D^k \mathrm{Rm}\|^2 + C \cdot \|D^k \mathrm{Rm}\| \cdot \|D^{k+1} \mathrm{Rm}\|. \qquad (4.36)
\end{aligned}
$$

One can then check that (since $m \leq k$)

$$
\begin{aligned}
&2\mathrm{Re}\left\langle \Delta_R(\nabla^m \overline{\nabla}^l \mathrm{Rm}), \nabla^m \overline{\nabla}^l \mathrm{Rm}\right\rangle \\
&= \left\langle \Delta_R(\nabla^m \overline{\nabla}^l \mathrm{Rm}), \nabla^m \overline{\nabla}^l \mathrm{Rm}\right\rangle + \left\langle \nabla^m \overline{\nabla}^l \mathrm{Rm}, \Delta_R(\nabla^m \overline{\nabla}^l \mathrm{Rm})\right\rangle \\
&= \Delta_R\big(\|\nabla^m \overline{\nabla}^l \mathrm{Rm}\|^2\big) - 2\|\nabla^{m+1} \overline{\nabla}^l \mathrm{Rm}\|^2 - 2\|\nabla^m \overline{\nabla}^{l+1} \mathrm{Rm}\|^2 \\
&\quad - 2\Big(\|\overline{\nabla} \nabla^m \overline{\nabla}^l \mathrm{Rm}\|^2 - \|\nabla^m \overline{\nabla}^{l+1} \mathrm{Rm}\|^2\Big) \\
&\leq \Delta_R\big(\|\nabla^m \overline{\nabla}^l \mathrm{Rm}\|^2\big) - 2\|\nabla^{m+1} \overline{\nabla}^l \mathrm{Rm}\|^2 - 2\|\nabla^m \overline{\nabla}^{l+1} \mathrm{Rm}\|^2 \\
&\quad + \sum_{i=0}^{m-1} C \cdot \|\nabla^m \overline{\nabla}^{l+1} \mathrm{Rm}\| \cdot \|\nabla^i \mathrm{Rm}\| \cdot \|\nabla^{m-1-i} \overline{\nabla}^l \mathrm{Rm}\| \\
&\quad + \sum_{i=0}^{m-1} C \cdot \|\nabla^i \mathrm{Rm}\|^2 \cdot \|\nabla^{m-1-i} \overline{\nabla}^l \mathrm{Rm}\|^2 \\
&\leq \Delta_R\big(\|\nabla^m \overline{\nabla}^l \mathrm{Rm}\|^2\big) - 2\|\nabla^{m+1} \overline{\nabla}^l \mathrm{Rm}\|^2 - 2\|\nabla^m \overline{\nabla}^{l+1} \mathrm{Rm}\|^2 \\
&\quad + C + C \cdot \|D^{k+1} \mathrm{Rm}\|. \qquad (4.37)
\end{aligned}
$$

Combining these together, we get

$$
\begin{aligned}
(\mathbf{VII}) \leq \frac{1}{2}\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot \Delta_R\big(\|\nabla^m\overline{\nabla}^l\mathrm{Rm}\|^2\big) \\
&- B^{-1}\cdot\|\nabla^{m+1}\overline{\nabla}^l\mathrm{Rm}\|^2 - B^{-1}\cdot\|\nabla^m\overline{\nabla}^{l+1}\mathrm{Rm}\|^2 \\
&+ C + C\cdot\|D^k\mathrm{Rm}\| + C\cdot\|D^{k+1}\mathrm{Rm}\| \\
&+ C\cdot\|D^k\mathrm{Rm}\|^2 + C\cdot\|D^k\mathrm{Rm}\|\cdot\|D^{k+1}\mathrm{Rm}\|. \qquad (4.38)
\end{aligned}
$$

**Term (VIII)** This final term has "too many" non-Laplacian covariant derivatives. In order to deal with this, we rewrite them in preparation for integration by parts and application of the Divergence Theorem. We keep track of the constant in front of the terms that are quadratic in the highest-order since we will want to cancel them out later.

Using the Commutator Identity (A.18), we have that

$$
\begin{aligned}
\nabla^m\overline{\nabla}^l\big(\nabla\overline{\nabla}(\mathrm{Rm}*\mathrm{Rm})\big) &= \overline{\nabla}\nabla^{m+1}\overline{\nabla}^l(\mathrm{Rm}*\mathrm{Rm}) \\
&+ \sum_{i=0}^{m}\sum_{j=0}^{l}\big(\nabla^{m-i}\overline{\nabla}^{l-j}(\mathrm{Rm}*\mathrm{Rm})\big)*(\nabla^i\overline{\nabla}^j\mathrm{Rm}). \qquad (4.39)
\end{aligned}
$$

From this, we see that

$$
\begin{aligned}
&\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot 2\alpha'\mathrm{Re}\Big\langle\nabla^m\overline{\nabla}^l\big(\nabla\overline{\nabla}(\mathrm{Rm}*\mathrm{Rm})\big), \nabla^m\overline{\nabla}^l\mathrm{Rm}\Big\rangle \\
&\leq \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot 2\alpha'\mathrm{Re}\,\langle\overline{\nabla}\nabla^{m+1}\overline{\nabla}^l(\mathrm{Rm}*\mathrm{Rm}), \nabla^m\overline{\nabla}^l\mathrm{Rm}\rangle \\
&\quad + \sum_{i=0}^{m}\sum_{j=0}^{l} C\cdot\|\nabla^{m-i}\overline{\nabla}^{l-j}(\mathrm{Rm}*\mathrm{Rm})\|\cdot\|\nabla^i\overline{\nabla}^j\mathrm{Rm}\|\cdot\|\nabla^m\overline{\nabla}^l\mathrm{Rm}\| \\
&\leq \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot 2\alpha'\mathrm{Re}\Big(\overline{\nabla}_{\bar{j}}\langle\nabla^{m+1}\overline{\nabla}^l(\mathrm{Rm}*\mathrm{Rm}), \nabla^m\overline{\nabla}^l\mathrm{Rm}\rangle^{\bar{j}}\Big) \\
&\quad - \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot 2\alpha'\mathrm{Re}\,\langle\nabla^{m+1}\overline{\nabla}^l(\mathrm{Rm}*\mathrm{Rm}), \nabla^{m+1}\overline{\nabla}^l\mathrm{Rm}\rangle \\
&\quad + C\cdot\|D^k\mathrm{Rm}\| + C\cdot\|D^k\mathrm{Rm}\|^2 \\
&\leq \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot 2\alpha'\mathrm{Re}\Big(\overline{\nabla}_{\bar{j}}\langle\nabla^{m+1}\overline{\nabla}^l(\mathrm{Rm}*\mathrm{Rm}), \nabla^m\overline{\nabla}^l\mathrm{Rm}\rangle^{\bar{j}}\Big) \\
&\quad + C\cdot\|D^k\mathrm{Rm}\| + C\cdot\|D^{k+1}\mathrm{Rm}\| \\
&\quad + C\cdot\|D^k\mathrm{Rm}\|^2 + C\cdot\|D^k\mathrm{Rm}\|\cdot\|D^{k+1}\mathrm{Rm}\| \\
&\quad + 4a_0BC_0\alpha'\|\nabla^{m+1}\overline{\nabla}^l\mathrm{Rm}\|^2, \qquad (4.40)
\end{aligned}
$$

where $a_0 > 1$ is a fixed predetermined constant independent of $k$ arising from the linear combinations obscured by the $*$ contraction notation in (4.6) and (4.7).

As such, we get

$$\begin{aligned}(\mathbf{VIII}) \leq\ & 2\alpha'\mathrm{Re}\left(\overline{\nabla}_{\bar{j}}\Big\langle\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot\big(\nabla^{m+1}\overline{\nabla}^l(\mathrm{Rm}*\mathrm{Rm})\big),\nabla^m\overline{\nabla}^l\mathrm{Rm}\Big\rangle^{\bar{j}}\right)\\ &+ C\cdot\|D^k\mathrm{Rm}\| + C\cdot\|D^{k+1}\mathrm{Rm}\|\\ &+ C\cdot\|D^k\mathrm{Rm}\|^2 + C\cdot\|D^k\mathrm{Rm}\|\cdot\|D^{k+1}\mathrm{Rm}\|\\ &+ 4a_0BC_0\alpha'\|\nabla^{m+1}\overline{\nabla}^l\mathrm{Rm}\|^2.\end{aligned} \tag{4.41}$$

**Combining the Terms** If we combine what we have from the terms **(I)** - **(VIII)**, we get the pointwise estimate that

$$\begin{aligned}&\frac{\partial}{\partial t}\|\nabla^m\overline{\nabla}^l\mathrm{Rm}\|^2\\ &\leq \frac{1}{2}\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot\Delta_R\big(\|\nabla^m\overline{\nabla}^l\mathrm{Rm}\|^2\big)\\ &\quad - B^{-1}\cdot\|\nabla^{m+1}\overline{\nabla}^l\mathrm{Rm}\|^2 - B^{-1}\cdot\|\nabla^m\overline{\nabla}^{l+1}\mathrm{Rm}\|^2\\ &\quad + 2\alpha'\mathrm{Re}\left(\overline{\nabla}_{\bar{j}}\Big\langle\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot\big(\nabla^{m+1}\overline{\nabla}^l(\mathrm{Rm}*\mathrm{Rm})\big),\nabla^m\overline{\nabla}^l\mathrm{Rm}\Big\rangle^{\bar{j}}\right)\\ &\quad + C + C\cdot\|D^k\mathrm{Rm}\| + C\cdot\|D^{k+1}\mathrm{Rm}\|\\ &\quad + C\cdot\|D^k\mathrm{Rm}\|^2 + C\cdot\|D^k\mathrm{Rm}\|\cdot\|D^{k+1}T\|\\ &\quad + C\cdot\|D^k\mathrm{Rm}\|\cdot\|D^{k+1}\mathrm{Rm}\| + C\cdot\|D^k\mathrm{Rm}\|\cdot\|D^{k+2}T\|\\ &\quad + 4a_0BC_0\alpha'\|\nabla^{m+1}\overline{\nabla}^l\mathrm{Rm}\|^2.\end{aligned} \tag{4.42}$$

We have the inequality

$$\begin{aligned}\|D^{k+1}\mathrm{Rm}\|^2 &= \sum_{a+b=k+1}\|\nabla^a\overline{\nabla}^b\mathrm{Rm}\|^2\\ &\leq \sum_{m+l=k}\|\nabla^{m+1}\overline{\nabla}^l\mathrm{Rm}\|^2 + \sum_{m+l=k}\|\nabla^m\overline{\nabla}^{l+1}\mathrm{Rm}\|^2,\end{aligned} \tag{4.43}$$

Summing over $m + l = k$ and using the Peter–Paul version of Young's Inequality

$$ab \leq \frac{1}{2}\epsilon^{-1}a^2 + \frac{1}{2}\epsilon b^2 \text{ for } \epsilon > 0, \tag{4.44}$$

we get that for $0 < \epsilon < 1$,

$$
\begin{aligned}
&\frac{\partial}{\partial t}\|D^k\mathrm{Rm}\|^2\\
&\leq \frac{1}{2}\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot\Delta_R\big(\|D^k\mathrm{Rm}\|^2\big) - B^{-1}\cdot\|D^{k+1}\mathrm{Rm}\|^2\\
&\quad + \sum_{m+l=k} 2\alpha'\mathrm{Re}\left(\overline{\nabla}_{\bar{j}}\Big\langle\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot\big(\nabla^{m+1}\overline{\nabla}^l(\mathrm{Rm}*\mathrm{Rm})\big), \nabla^m\overline{\nabla}^l\mathrm{Rm}\Big\rangle^{\bar{j}}\right)\\
&\quad + C\epsilon^{-1}\cdot\Big(1 + \|D^k\mathrm{Rm}\|^2 + \|D^{k+1}T\|^2\Big)\\
&\quad + \Big[C\epsilon + 4a_0BC_0\alpha'\Big]\cdot\|D^{k+1}\mathrm{Rm}\|^2 + C\epsilon\cdot\|D^{k+2}T\|^2.
\end{aligned} \tag{4.45}
$$

A similar treatment for covariant derivatives of $T$ yields

$$
\begin{aligned}
&\frac{\partial}{\partial t}\|D^{k+1}T\|^2\\
&\leq \frac{1}{2}\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot\Delta_R\big(\|D^{k+1}T\|^2\big) - B^{-1}\cdot\|D^{k+2}T\|^2\\
&\quad + \sum_{m'+l'=k+1} 2\alpha'\mathrm{Re}\left(\nabla_i\Big\langle\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot\big(\nabla^{m'}\overline{\nabla}^{l'}(\mathrm{Rm}*\mathrm{Rm})\big), \nabla^{m'}\overline{\nabla}^{l'}T\Big\rangle^{i}\right)\\
&\quad + C\epsilon^{-1}\cdot\Big(1 + \|D^k\mathrm{Rm}\|^2 + \|D^{k+1}T\|^2\Big)\\
&\quad + \Big[C\epsilon + 2a_0BC_0\alpha'\Big]\cdot\Big(\|D^{k+1}\mathrm{Rm}\|^2 + \|D^{k+2}T\|^2\Big).
\end{aligned} \tag{4.46}
$$

**$L^{2p}$-Estimates**

We now define a convenient test function by setting

$$G_q = \|D^q\mathrm{Rm}\|^2 + \|D^{q+1}T\|^2. \tag{4.47}$$

Summing together (4.45) and (4.46), we get

$$
\begin{aligned}
\frac{\partial}{\partial t}G_k &\leq \frac{1}{2}\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot\Delta_R G_k - B^{-1}\cdot G_{k+1}\\
&\quad + \sum_{m+l=k} 2\alpha'\mathrm{Re}\left(\overline{\nabla}_{\bar{j}}\Big\langle\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot\big(\nabla^{m+1}\overline{\nabla}^l(\mathrm{Rm}*\mathrm{Rm})\big), \nabla^m\overline{\nabla}^l\mathrm{Rm}\Big\rangle^{\bar{j}}\right)\\
&\quad + \sum_{m'+l'=k+1} 2\alpha'\mathrm{Re}\left(\nabla_i\Big\langle\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot\big(\nabla^{m'}\overline{\nabla}^{l'}(\mathrm{Rm}*\mathrm{Rm})\big), \nabla^{m'}\overline{\nabla}^{l'}T\Big\rangle^{i}\right)
\end{aligned}
$$

$$+ C\epsilon^{-1} \cdot \Big(1 + G_k\Big) + \Big[C\epsilon + 6a_0 BC_0\alpha'\Big] \cdot G_{k+1}. \tag{4.48}$$

By integrating and using the above, we see that for $p \geq 3$,

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\int_X G_k^p\right) \leq \int_X \left(\frac{\partial}{\partial t} G_k^p\right) + C \cdot \int_X G_k^p \\
&\leq p \cdot \int_X G_k^{p-1} \cdot \left(\frac{\partial}{\partial t} G_k\right) + C \cdot \int_X G_k^p \\
&\leq \frac{p}{2} \cdot \int_X \left(\frac{1}{2\|\Upsilon\|_\omega}\right) \cdot G_k^{p-1} \cdot (\Delta_R G_k) - B^{-1} p \cdot \int_X G_k^{p-1} \cdot G_{k+1} \\
&\quad + \sum_{m+l=k} 2\alpha' p \mathrm{Re}\left(\int_X G_k^{p-1}\right. \\
&\qquad\qquad \left. \cdot \overline{\nabla}_{\bar{j}} \Big\langle \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot \big(\nabla^{m+1}\overline{\nabla}^l (\mathrm{Rm} * \mathrm{Rm})\big), \nabla^m \overline{\nabla}^l \mathrm{Rm} \Big\rangle^{\bar{j}} \right) \\
&\quad + \sum_{m'+l'=k+1} 2\alpha' p \mathrm{Re}\left(\int_X G_k^{p-1}\right. \\
&\qquad\qquad \left. \cdot \nabla_i \Big\langle \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot \big(\nabla^{m'}\overline{\nabla}^{l'} (\mathrm{Rm} * \mathrm{Rm})\big), \nabla^{m'} \overline{\nabla}^{l'} T \Big\rangle^{i} \right) \\
&\quad + C\epsilon^{-1} \cdot \int_X G_k^{p-1} \cdot \Big(1 + G_k\Big) + \Big[C\epsilon + 6a_0 BC_0 \alpha' p\Big] \cdot \int_X G_k^p \cdot G_{k+1},
\end{aligned} \tag{4.49}$$

where the generic constant $C$ in each line may now also depend on $p$. The second term in the first inequality comes from the evolving volume form and the fact that $\|\frac{\partial}{\partial t} g\|$ is bounded along the flow (see Remark 4.2.1).

**Remark 4.2.2.** We impose the $p \geq 3$ condition for now to avoid potentially dividing by 0 in the future. This is required since some intermediary steps in our future equations involve terms of the form $G_k^{p-3}$ (see the penultimate inequality in (4.53)). The set where $G_k$ vanishes cannot naïvely be removed as it would result in a boundary component after integration by parts. The case where $p \in [1, 3)$ will be addressed later.

Using the identity

$$\Delta_R G_k^p = 2p(p-1) \cdot G_k^{p-2} \cdot \|\nabla G_k\|^2 + p \cdot G_k^{p-1} \cdot (\Delta_R G_k), \tag{4.50}$$

we can write the above with an extra negative term.

$$\begin{aligned}
\frac{\partial}{\partial t}\bigg(\int_X G_k^p\bigg) &\leq \frac{1}{2}\cdot\int_X\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot(\Delta_R G_k^p) \\
&- B^{-1}p(p-1)\cdot\int_X G_k^{p-2}\cdot\|\nabla G_k\|^2 - B^{-1}p\cdot\int_X G_k^{p-1}\cdot G_{k+1} \\
&+ \sum_{m+l=k} 2\alpha' p\mathrm{Re}\bigg(\int_X G_k^{p-1} \\
&\qquad \cdot\overline{\nabla}_{\bar{j}}\Big\langle\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot\big(\nabla^{m+1}\overline{\nabla}^l(\mathrm{Rm}*\mathrm{Rm})\big),\nabla^m\overline{\nabla}^l\mathrm{Rm}\Big\rangle^{\bar{j}}\bigg) \\
&+ \sum_{m'+l'=k+1} 2\alpha' p\mathrm{Re}\bigg(\int_X G_k^{p-1} \\
&\qquad \cdot\nabla_i\Big\langle\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot\big(\nabla^{m'}\overline{\nabla}^{l'}(\mathrm{Rm}*\mathrm{Rm})\big),\nabla^{m'}\overline{\nabla}^{l'}T\Big\rangle^{i}\bigg) \\
&+ C\epsilon^{-1}\cdot\int_X G_k^{p-1}\cdot\Big(1+G_k\Big) + \Big[C\epsilon + 6a_0 BC_0\alpha' p\Big]\cdot\int_X G_k^p\cdot G_{k+1}.
\end{aligned} \tag{4.51}$$

By using integration by parts and the Divergence Theorem (A.19), we can absorb the Laplacian term and divergence terms into the negative ones.

First, we see that

$$\begin{aligned}
&\frac{1}{2}\cdot\int_X\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot(\Delta_R G_k^p) \\
&= \frac{1}{2}\int_X\nabla_i\Big[\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot g^{i\bar{j}}\cdot\overline{\nabla}_{\bar{j}}G_k^p\Big] - \frac{1}{2}\int_X g^{i\bar{j}}\cdot\nabla_i\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot\overline{\nabla}_{\bar{j}}G_k^p \\
&\quad + \frac{1}{2}\int_X\overline{\nabla}_{\bar{j}}\Big[\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot g^{i\bar{j}}\cdot\nabla_i G_k^p\Big] - \frac{1}{2}\int_X g^{i\bar{j}}\cdot\overline{\nabla}_{\bar{j}}\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot\nabla_i G_k^p \\
&\leq C\cdot\int_X\|\nabla G_k^p\| \\
&\leq C\cdot\int_X G_k^{p-1}\cdot\|\nabla G_k\| \\
&\leq C\epsilon^{-1}\cdot\int_X G_k^{p-2}\cdot G_k^2 + C\epsilon\cdot\int_X G_k^{p-2}\cdot\|\nabla G_k\|^2 \\
&= C\epsilon^{-1}\cdot\int_X G_k^p + C\epsilon\cdot\int_X G_k^{p-2}\cdot\|\nabla G_k\|^2.
\end{aligned} \tag{4.52}$$

Next, using the Divergence Theorem (A.19) and that $\|D^k \mathrm{Rm}\|^2 \leq G_k$, we can write

$$
\begin{aligned}
&\sum_{m+l=k} 2\alpha' p \mathrm{Re}\Bigg(\int_X G_k^{p-1} \\
&\qquad \cdot \overline{\nabla}_{\bar{j}}\Big\langle \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot \big(\nabla^{m+1}\overline{\nabla}^l(\mathrm{Rm} * \mathrm{Rm})\big), \nabla^m \overline{\nabla}^l \mathrm{Rm}\Big\rangle^{\bar{j}}\Bigg) \\
&= \sum_{m+l=k} 2\alpha' p \mathrm{Re}\Bigg(\int_X \overline{\nabla}_{\bar{j}}\Big\langle G_k^{p-1} \cdot \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \\
&\qquad \cdot \big(\nabla^{m+1}\overline{\nabla}^l(\mathrm{Rm} * \mathrm{Rm})\big), \nabla^m \overline{\nabla}^l \mathrm{Rm}\Big\rangle^{\bar{j}}\Bigg) \\
&\quad - \sum_{m+l=k} 2\alpha' p \mathrm{Re}\Bigg(\int_X \overline{\nabla}_{\bar{j}} G_k^{p-1} \\
&\qquad \cdot \Big\langle \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot \big(\nabla^{m+1}\overline{\nabla}^l(\mathrm{Rm} * \mathrm{Rm})\big), \nabla^m \overline{\nabla}^l \mathrm{Rm}\Big\rangle^{\bar{j}}\Bigg) \\
&\leq C \cdot \int_X G_k^{p-1} \cdot \|D^k \mathrm{Rm}\| \cdot \|D^{k+1}(\mathrm{Rm} * \mathrm{Rm})\| \\
&\quad + \sum_{m+l=k} 2a_0 B\alpha' p(p-1) \\
&\quad \cdot \int_X G_k^{p-2} \cdot \|\overline{\nabla} G_k\| \cdot \|\nabla^m \overline{\nabla}^l \mathrm{Rm}\| \cdot \|\nabla^{m+1}\overline{\nabla}^l(\mathrm{Rm} * \mathrm{Rm})\| \\
&\leq C \cdot \int_X G_k^{p-1} \cdot \|D^k \mathrm{Rm}\| + C \cdot \int_X G_k^{p-1} \cdot \|D^k \mathrm{Rm}\|^2 \\
&\quad + C \cdot \int_X G_k^{p-1} \cdot \|D^k \mathrm{Rm}\| \cdot \|D^{k+1} \mathrm{Rm}\| \\
&\quad + C \cdot \int_X G_k^{p-2} \cdot \|\overline{\nabla} G_k\| \cdot \|D^k \mathrm{Rm}\| + C \cdot \int_X G_k^{p-2} \cdot \|\overline{\nabla} G_k\| \cdot \|D^k \mathrm{Rm}\|^2 \\
&\quad + \sum_{m+l=k} 4a_0 B C_0 \alpha' p(p-1) \\
&\quad \cdot \int_X G_k^{p-2} \cdot \|\overline{\nabla} G_k\| \cdot \|\nabla^m \overline{\nabla}^l \mathrm{Rm}\| \cdot \|\nabla^{m+1}\overline{\nabla}^l \mathrm{Rm}\| \\
&\leq C\epsilon^{-1} \cdot \int_X G_k^{p-1} + C\epsilon^{-1} \cdot \int_X G_k^p \\
&\quad + C\epsilon \cdot \int_X G_k^{p-1} \cdot \|D^{k+1} \mathrm{Rm}\|^2 + C\epsilon \cdot \int_X G_k^{p-2} \cdot \|\overline{\nabla} G_k\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{m+l=k} 2a_0 B C_0 \alpha' p(p-1) \cdot \int_X G_k^{p-3} \cdot \|\overline{\nabla} G_k\|^2 \cdot \|\nabla^m \overline{\nabla}^l \mathrm{Rm}\|^2 \\
&+ \sum_{m+l=k} 2a_0 B C_0 \alpha' p(p-1) \cdot \int_X G_k^{p-3} \cdot G_k^2 \cdot \|\nabla^{m+1} \overline{\nabla}^l \mathrm{Rm}\|^2 \\
\leq\ & C\epsilon^{-1} \cdot \int_X G_k^{p-1} \cdot \Big(1 + G_k\Big) \\
&+ \Big[C\epsilon + 2a_0 B C_0 \alpha' p(p-1)\Big] \cdot \int_X G_k^{p-1} \cdot \|D^{k+1} \mathrm{Rm}\|^2 \\
&+ \Big[C\epsilon + 2a_0 B C_0 \alpha' p(p-1)\Big] \cdot \int_X G_k^{p-2} \cdot \|\overline{\nabla} G_k\|^2. \qquad (4.53)
\end{aligned}
$$

Similar methods show that the other divergence term in (4.51) has the same upper bound. Substituting these in, and using that

$$
p \cdot G_k^{p-1} \leq (p-1) \cdot G_k^p + 1, \qquad (4.54)
$$

we get

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\int_X G_k^p\right) &\leq C\epsilon^{-1} \cdot \int_X \Big(1 + G_k^p\Big) \\
&+ \Big[C\epsilon + 4a_0 B C_0 \alpha' p\Big(p + \frac{1}{2}\Big) - B^{-1} p\Big] \cdot \int_X G_k^{p-1} \cdot G_{k+1} \\
&+ \Big[C\epsilon + 4a_0 B C_0 \alpha' p(p-1) - B^{-1} p(p-1)\Big] \cdot \int_X G_k^{p-2} \cdot \|\overline{\nabla} G_k\|^2. \qquad (4.55)
\end{aligned}
$$

As such, if

$$
\alpha' < \frac{1}{4a_0 B^2 C_0 (p + \frac{1}{2})}, \qquad (4.56)
$$

then by choosing $\epsilon = \epsilon(k, \alpha', p)$ carefully, we can absorb corresponding terms into the negative ones. This leaves

$$
\frac{\partial}{\partial t}\left(\int_X G_k^p\right) \leq C + C \cdot \int_X G_k^p, \qquad (4.57)
$$

where we have again used that the volume is bounded along the flow. Recall Grönwall's Inequality, which states that if

$$
u'(t) \leq \beta(t) \cdot u(t) \text{ on } (a, b),
$$

then

$$u(t) \leq u(a) \cdot \exp\left(\int_a^t \beta(s)\, ds\right) \text{ on } [a, b).$$

By setting $u = 1 + \int_X G_k^p$ and $\beta = C$ and applying Grönwall's Inequality, we have

**Theorem 4.2.3.** *Let $k \geq 2$ and $p \geq 3$. Set $G_k = \|D^k \mathrm{Rm}\|^2 + \|D^{k+1} T\|^2$. Given the assumptions (4.14) - (4.16), if*

$$\alpha' < \frac{1}{4a_0 B^2 C_0 (p + \frac{1}{2})}, \tag{4.58}$$

*then there exists some constant $\Lambda_p = \Lambda_p(k, \alpha') > 0$ such that*

$$\left(\int_X G_k^p(t)\right) \leq \left(1 + \int_X G_k^p(0)\right) \cdot e^{\Lambda_p t} < \left(1 + \int_X G_k^p(0)\right) \cdot e^{\Lambda_p \tau}. \tag{4.59}$$

*That is, $\int_X G_k^p(t)$ is uniformly bounded along the flow.*

*In particular, after taking a 2p-th root, we get that both*

$$\left(\int_X \|D^k \mathrm{Rm}(t)\|^{2p}\right)^{\frac{1}{2p}} \text{ and } \left(\int_X \|D^{k+1} T(t)\|^{2p}\right)^{\frac{1}{2p}} \tag{4.60}$$

*are uniformly bounded along the Anomaly flow.*

Using the uniform boundedness of the volume, we can retrieve the $L^{2p}$-bounds for $p \in [1, 3)$ via Hölder's Inequality.

**Corollary 4.2.4.** *Let $k \geq 2$ and $p \in [1, 3)$. Set $G_k = \|D^k \mathrm{Rm}\|^2 + \|D^{k+1} T\|^2$. Given the assumptions (4.14) - (4.16), if*

$$\alpha' < \frac{1}{14 a_0 B^2 C_0}, \tag{4.61}$$

*then $\int_X G_k^p(t)$ and both*

$$\left(\int_X \|D^k \mathrm{Rm}(t)\|^{2p}\right)^{\frac{1}{2p}} \text{ and } \left(\int_X \|D^{k+1} T(t)\|^{2p}\right)^{\frac{1}{2p}} \tag{4.62}$$

*are uniformly bounded along the Anomaly flow.*

### 4.2.3 Estimates on $\|D^{k+1}\mathrm{Rm}\|$ and $\|D^{k+2}T\|$

As mentioned earlier, we require $L^\infty$-bounds instead of the weaker $L^{2p}$-bounds from the previous section. To achieve this, we appeal to an argument of Hamilton [Ham82] and first find $L^{2p}$-bounds on higher-order derivatives under the same initial assumptions. This will allow us to appeal to the Sobolev Embedding Theorem in order to recover $L^\infty$-bounds.

For this section, we assume that $m + l = k \geq 3$. As in §4.2.2, this is to ensure that terms are at most quadratic in unknowns. The other cases are deferred to §4.3.1 and §4.3.2.

Recall the test function $G_q$ defined in (4.47). In an analogous manner to §4.2.2, we can check that under the assumptions (4.14) - (4.16), we get the pointwise inequality

$$\begin{aligned}
\frac{\partial}{\partial t}G_{k+1} \leq \frac{1}{2}\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot\Delta_R G_{k+1} - B^{-1}\cdot G_{k+2} \\
+ \sum_{m+l=k+1} 2\alpha'\mathrm{Re}\Big(\overline{\nabla}_{\bar{j}}\Big\langle\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot\big(\nabla^{m+1}\overline{\nabla}^l(\mathrm{Rm}*\mathrm{Rm})\big), \nabla^m\overline{\nabla}^l\mathrm{Rm}\Big\rangle^{\bar{j}}\Big) \\
+ \sum_{m'+l'=k+2} 2\alpha'\mathrm{Re}\Big(\nabla_i\Big\langle\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot\big(\nabla^{m'}\overline{\nabla}^{l'}(\mathrm{Rm}*\mathrm{Rm})\big), \nabla^{m'}\overline{\nabla}^{l'}T\Big\rangle^{i}\Big) \\
+ C(\epsilon')^{-1}\cdot\Big(1 + G_k + G_{k+1}\Big) + \Big[C\epsilon' + 6a_0BC_0\alpha'\Big]\cdot G_{k+2}
\end{aligned} \tag{4.63}$$

for $0 < \epsilon' < 1$.

Integration shows that for $p \geq 3$,

$$\begin{aligned}
\frac{\partial}{\partial t}\Big(\int_X G_{k+1}^p\Big) \leq \frac{1}{2}\cdot\int_X\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot(\Delta_R G_{k+1}^p) \\
- B^{-1}p(p-1)\cdot\int_X G_{k+1}^{p-2}\cdot\|\nabla G_{k+1}\|^2 - B^{-1}p\cdot\int_X G_{k+1}^{p-1}\cdot G_{k+2} \\
+ \sum_{m+l=k+1} 2\alpha' p\mathrm{Re}\Bigg(\int_X G_{k+1}^{p-1} \\
\cdot\overline{\nabla}_{\bar{j}}\Big\langle\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot\big(\nabla^{m+1}\overline{\nabla}^l(\mathrm{Rm}*\mathrm{Rm})\big), \nabla^m\overline{\nabla}^l\mathrm{Rm}\Big\rangle^{\bar{j}}\Bigg)
\end{aligned}$$

$$+ \sum_{m'+l'=k+2} 2\alpha' p \mathrm{Re}\Bigg( \int_X G_{k+1}^{p-1} \cdot \nabla_i \Big\langle \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot \big(\nabla^{m'}\overline{\nabla}^{l'}(\mathrm{Rm} * \mathrm{Rm})\big), \nabla^{m'}\overline{\nabla}^{l'} T \Big\rangle^i \Bigg)$$
$$+ C(\epsilon')^{-1} \cdot \int_X G_{k+1}^{p-1} \cdot \Big(1 + G_k + G_{k+1}\Big)$$
$$+ \Big[C\epsilon' + 6a_0 B C_0 \alpha' p\Big] \cdot \int_X G_{k+1}^p \cdot G_{k+2}. \tag{4.64}$$

As in the previous section, the Laplacian term can be absorbed by the other terms

$$\frac{1}{2} \cdot \int_X \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot (\Delta_R G_{k+1}^p) \leq C(\epsilon')^{-1} \cdot \int_X G_{k+1}^p + C\epsilon' \cdot \int_X G_{k+1}^{p-2} \cdot \|\nabla G_{k+1}\|^2. \tag{4.65}$$

Similar computations show that the divergence terms can together be bound above by

$$C(\epsilon')^{-1} \cdot \int_X G_{k+1}^{p-1} \cdot \Big(1 + G_k + G_{k+1}\Big)$$
$$+ \Big[C\epsilon' + 4a_0 B C_0 \alpha' p(p-1)\Big] \cdot \int_X G_k^{p-1} \cdot G_{k+2}$$
$$+ \Big[C\epsilon' + 4a_0 B C_0 \alpha' p(p-1)\Big] \cdot \int_X G_{k+1}^{p-2} \cdot \|\overline{\nabla} G_{k+1}\|^2. \tag{4.66}$$

In tandem with (4.64), we have

$$\frac{\partial}{\partial t}\Bigg(\int_X G_{k+1}^p\Bigg) \leq C(\epsilon')^{-1} \cdot \int_X \Big(1 + G_k^p + G_{k+1}^p\Big)$$
$$+ \Big[C\epsilon' + 4a_0 B C_0 \alpha' p\Big(p + \frac{1}{2}\Big) - B^{-1} p\Big] \cdot \int_X G_{k+1}^{p-1} \cdot G_{k+2}$$
$$+ \Big[C\epsilon' + 4a_0 B C_0 \alpha' p(p-1) - B^{-1} p(p-1)\Big] \cdot \int_X G_{k+1}^{p-2} \cdot \|\overline{\nabla} G_{k+1}\|^2, \tag{4.67}$$

where we have also used that by Young's Inequality

$$p \cdot G_{k+1}^{p-1} \cdot G_k \leq (p-1) \cdot G_{k+1}^p + G_k^p, \tag{4.68}$$

and

$$p \cdot G_{k+1}^{p-1} \leq (p-1) \cdot G_{k+1}^{p} + 1. \tag{4.69}$$

If $\alpha'$ is sufficiently small, then $\epsilon' = \epsilon'(k, \alpha', p)$ can be chosen such that

$$\frac{\partial}{\partial t}\left(\int_X G_{k+1}^p\right) \leq C \cdot \int_X 1 + C \cdot \int_X G_k^p + C \cdot \int_X G_{k+1}^p. \tag{4.70}$$

By our assumptions, the volume is uniformly bounded along the flow. Further, by Theorem 4.2.3 and Corollary 4.2.4, so is $\int_X G_k^p$. We conclude the following:

**Corollary 4.2.5.** *Let $k \geq 3$ and $G_{k+1} = \|D^{k+1}\mathrm{Rm}\|^2 + \|D^{k+2}T\|^2$. Suppose the assumptions (4.14) - (4.16) hold.*

*i) If $p \geq 3$ and*

$$\alpha' < \frac{1}{4a_0 B^2 C_0 (p + \frac{1}{2})}, \tag{4.71}$$

*then there exists some constant $\Lambda_p' = \Lambda_p'(k, \alpha') > 0$ such that*

$$\int_X G_{k+1}^p(t) \leq \left(1 + \int_X G_{k+1}^p(0)\right) e^{\Lambda_p' t} < \left(1 + \int_X G_{k+1}^p(0)\right) e^{\Lambda_p' \tau}. \tag{4.72}$$

*That is, $\int_X G_{k+1}^p(t)$ is uniformly bounded along the flow.*

*In particular, we get that both*

$$\left(\int_X \|D^{k+1}\mathrm{Rm}(t)\|^{2p}\right)^{\frac{1}{2p}} \quad \text{and} \quad \left(\int_X \|D^{k+2}T(t)\|^{2p}\right)^{\frac{1}{2p}} \tag{4.73}$$

*are bounded along the Anomaly flow;*

*ii) If instead $p \in [1, 3)$ and*

$$\alpha' < \frac{1}{14 a_0 B^2 C_0}, \tag{4.74}$$

*then $\int_X G_{k+1}^p(t)$ and both*

$$\left(\int_X \|D^{k+1}\mathrm{Rm}(t)\|^{2p}\right)^{\frac{1}{2p}} \quad \text{and} \quad \left(\int_X \|D^{k+2}T(t)\|^{2p}\right)^{\frac{1}{2p}} \tag{4.75}$$

*are uniformly bounded along the Anomaly flow.*

### 4.2.4 Sobolev Embedding and Induction

Recall our base assumptions from §4.2.1: for some $k \geq 1$ there exist positive constants $B, C_0, C_1, \ldots, C_{k-1}$ such that along the Anomaly flow on $[0, \tau)$

$$B^{-1} \leq \left(\frac{1}{2\|\Upsilon\|_\omega}\right) \leq B, \tag{4.76}$$

$$\|T\|, \|\overline{T}\|, \|\mathrm{Rm}\|, \|DT\|, \|D\overline{T}\| \leq C_0, \tag{4.77}$$

$$\|D^q\mathrm{Rm}\|, \|D^{q+1}T\|, \|D^{q+1}\overline{T}\| \leq C_q \text{ for } 1 \leq q \leq k-1. \tag{4.78}$$

Recall also that $a_0$ is a predetermined constant from our expressions (4.6) and (4.7) that is inherent to the Anomaly flow.

Theorem 4.2.3 and Corollaries 4.2.4 and 4.2.5 have shown that if $k \geq 3$, $p \geq 3$, and if

$$\alpha' < \frac{1}{4a_0 B^2 C_0 (p + \frac{1}{2})}, \tag{4.79}$$

then each of

$$\|D^k\mathrm{Rm}\|_{L^{2p}(X)}, \|D^{k+1}T\|_{L^{2p}(X)}, \|D^{k+1}\mathrm{Rm}\|_{L^{2p}(X)}, \|D^{k+2}T\|_{L^{2p}(X)} \tag{4.80}$$

are bounded uniformly in $t$ along the Anomaly flow.

Suppose this holds for some $p$ with $2p > n = 6$. The Sobolev Embedding Theorem on $D^k\mathrm{Rm}$ and $D^{k+1}T$ then provides $L^\infty$-bounds on $D^k\mathrm{Rm}$ and $D^{k+1}T$ uniform in $t$ along the flow. (See the paragraphs following Lemma 14.3 and also Lemma 14.4 of [Ham82] for more details.) That is, there exists some $C_k$ such that

$$\|D^k\mathrm{Rm}\|, \|D^{k+1}T\|, \|D^{k+1}\overline{T}\| \leq C_k. \tag{4.81}$$

Importantly, the condition (4.79) on the slope parameter $\alpha'$ does not depend on $k$ and so we can induct on $k$ to obtain $L^\infty$-bounds for $\|D^q\mathrm{Rm}\|$ and $\|D^{q+1}T\|$ for each $q \geq 3$. We thus have

**Corollary 4.2.6.** *Suppose that the assumptions* (4.14) - (4.16) *hold for* $k = 3$. *If*

$$\alpha' < \frac{1}{14 a_0 B^2 C_0}, \tag{4.82}$$

*then there exist positive constants* $C_q$ *for* $q \geq 3$ *such that*

$$\|D^q\mathrm{Rm}\|, \|D^{q+1}T\|, \|D^{q+1}\overline{T}\| \leq C_q \tag{4.83}$$

*along the Anomaly flow on* $t \in [0, \tau)$.

**Remark 4.2.7.** We note that the constants $C_q$ for $q \geq 3$ from the previous corollary depend on $\tau$, the initial metric $g_0$, the slope parameter $\alpha'$, and the initial bounds $B, C_0, C_1, C_2$.

## 4.3 Lowering the Base Assumptions

We now aim to lower the initial value of $k$ in Corollary 4.2.6. To achieve the $k = 2$ case, we only need to establish the higher-order estimates of §4.2.3. The $k = 1$ case requires more work since the estimates on $\|D^k \mathrm{Rm}\|$ and $\|D^{k+1} T\|$ also need to be reproven.

### 4.3.1 The $k = 2$ Case

The procedure in this case is fairly similar to before as we have the estimates and the result of Theorem 4.2.3 as a starting point.

Analogous computations to those in §4.2.2 show that

$$\begin{aligned}\frac{\partial}{\partial t} G_3 \leq \frac{1}{2}\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot (\Delta_R G_3) - B^{-1} \cdot G_4 \\ + \sum_{m+l=3} 2\alpha' \mathrm{Re}\Big(\nabla_{\bar{j}} \Big\langle \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot \big(\nabla^{m+1}\overline{\nabla}^l (\mathrm{Rm} * \mathrm{Rm})\big), \nabla^m \overline{\nabla}^l \mathrm{Rm} \Big\rangle^{\bar{j}}\Big) \\ + \sum_{m'+l'=4} 2\alpha' \mathrm{Re}\Big(\nabla_i \Big\langle \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot \big(\nabla^{m'}\overline{\nabla}^{l'} (\mathrm{Rm} * \mathrm{Rm})\big), \nabla^{m'} \overline{\nabla}^{l'} T \Big\rangle^{i}\Big) \\ + C(\epsilon') \cdot \Big(1 + G_2 + \|D^2 \mathrm{Rm}\|^4 + G_3\Big) + \Big[C\epsilon' + 6a_0 B C_0 \alpha'\Big] \cdot G_4.\end{aligned} \tag{4.84}$$

The main difference between this case and the $k \geq 3$ case is that the term

$$\begin{aligned}\langle D^4(\mathrm{Rm} * \mathrm{Rm}), D^4 \mathrm{Rm}\rangle \leq C \cdot \|D^4 \mathrm{Rm}\|^2 + C \cdot \|D^3 \mathrm{Rm}\| \cdot \|D^4 \mathrm{Rm}\| \\ + C \cdot \|D^2 \mathrm{Rm}\|^2 \cdot \|D^4 \mathrm{Rm}\|\end{aligned} \tag{4.85}$$

is bounded by a term that is cubic in the unknowns $\|D^2 \mathrm{Rm}\|$, $\|D^3 \mathrm{Rm}\|$, and $\|D^4 \mathrm{Rm}\|$. After applying Young's Inequality, the above can be bounded by

$$\begin{aligned}\langle D^4(\mathrm{Rm} * \mathrm{Rm}), D^4 \mathrm{Rm}\rangle \leq C(\epsilon')^{-1} \cdot \|D^2 \mathrm{Rm}\|^4 + C(\epsilon')^{-1} \cdot \|D^3 \mathrm{Rm}\|^2 \\ + C\epsilon' \cdot \|D^4 \mathrm{Rm}\|^2,\end{aligned} \tag{4.86}$$

which is how the $\|D^2 \mathrm{Rm}\|^4$ appears in (4.84).

Using that $\|D^2\mathrm{Rm}\|^4 \leq G_2^2$, we see that for $p \geq 3$,

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\int_X G_3^p\right) \leq \frac{1}{2}\cdot\int_X \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot(\Delta_R G_3^p) \\
& - B^{-1}p\cdot\int_X G_3^{p-1}\cdot G_4 - B^{-1}p(p-1)\cdot\int_X G_3^{p-2}\cdot\|\nabla G_3\|^2 \\
& + \sum_{m+l=3} 2\alpha' p\mathrm{Re}\Bigg(\int_X G_3^{p-1} \\
& \qquad \cdot \overline{\nabla}_{\bar{j}}\Big\langle\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot\big(\nabla^{m+1}\overline{\nabla}^l(\mathrm{Rm}*\mathrm{Rm})\big), \nabla^m\overline{\nabla}^l\mathrm{Rm}\Big\rangle^{\bar{j}}\Bigg) \\
& + \sum_{m'+l'=4} 2\alpha' p\mathrm{Re}\Bigg(\int_X G_3^{p-1} \\
& \qquad \cdot \nabla_i\Big\langle\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot\big(\nabla^{m'}\overline{\nabla}^{l'}(\mathrm{Rm}*\mathrm{Rm})\big), \nabla^{m'}\overline{\nabla}^{l'}T\Big\rangle^{i}\Bigg) \\
& + C(\epsilon')^{-1}\cdot\int_X G_3^{p-1}\cdot\Big(1+G_2+G_2^2+G_3\Big) \\
& + \Big[C\epsilon' + 6a_0BC_0\alpha' p\Big]\cdot\int_X G_3^{p-1}\cdot G_4. \qquad (4.87)
\end{aligned}
$$

The Laplacian term is again well-behaved for our purposes

$$
\begin{aligned}
& \frac{1}{2}\cdot\int_X\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot(\Delta_R G_3^p) \\
& \leq C(\epsilon')\cdot\int_X G_3^p + C\epsilon'\cdot\int_X G_3^{p-2}\cdot\|\nabla G_3\|^2. \qquad (4.88)
\end{aligned}
$$

Further, the inner product terms can be bounded by

$$
\begin{aligned}
& C(\epsilon')^{-1}\cdot\int_X G_3^{p-1}\cdot\Big(1+G_2^2+G_3\Big) \\
& + \Big[C\epsilon' + 4a_0BC_0\alpha' p(p-1)\Big]\cdot\int_X G_3^{p-1}\cdot\|D^4\mathrm{Rm}\|^2 \\
& + \Big[C\epsilon' + 4a_0BC_0\alpha' p(p-1)\Big]\cdot\int_X G_3^{p-2}\cdot\|\overline{\nabla}G_3\|^2. \qquad (4.89)
\end{aligned}
$$

As before, we can apply Young's Inequality to get that

$$\frac{\partial}{\partial t}\bigg(\int_X G_3^p\bigg) \leq C(\epsilon')^{-1}\cdot\int_X\Big(1+G_2^p+G_2^{2p}+G_3^p\Big)$$
$$+\Big[C\epsilon'+4a_0BC_0\alpha' p\Big(p+\frac{1}{2}\Big)-B^{-1}p\Big]\cdot\int_X G_3^{p-1}\cdot G_4$$
$$+\Big[C\epsilon'+4a_0BC_0\alpha' p(p-1)-B^{-1}p(p-1)\Big]\cdot\int_X G_3^{p-2}\cdot\|\nabla G_3\|^2. \quad (4.90)$$

Taking note that we also need $\int_X G_2^{2p}$ to be uniformly bounded, we then conclude the following from Theorem 4.2.3 and Corollaries 4.2.4 and 4.2.5:

**Corollary 4.3.1.** *Set* $k=2$ *and* $G_3=\|D^3\mathrm{Rm}\|^2+\|D^4\mathrm{Rm}\|^2$*. Suppose the assumptions* (4.14) - (4.16) *hold.*

*i) If* $p\geq 3$ *and*

$$\alpha' < \frac{1}{4a_0B^2C_0(2p+\frac{1}{2})}, \quad (4.91)$$

*then there exists some constant* $\Lambda_p'=\Lambda_p'(k,\alpha')>0$ *such that*

$$\int_X G_3^p(t)\leq\bigg(1+\int_X G_3^p(0)\bigg)e^{\Lambda_p' t}<\bigg(1+\int_X G_3^p(0)\bigg)e^{\Lambda_p'\tau}. \quad (4.92)$$

*That is,* $\int_X G_3^p(t)$ *is uniformly bounded along the flow.*

*In particular, we get that both*

$$\bigg(\int_X\|D^3\mathrm{Rm}(t)\|^{2p}\bigg)^{\frac{1}{2p}} \text{ and } \bigg(\int_X\|D^4T(t)\|^{2p}\bigg)^{\frac{1}{2p}} \quad (4.93)$$

*are bounded along the Anomaly flow;*

*ii) If instead* $p\in[1,3)$ *and*

$$\alpha'<\frac{1}{26a_0B^2C_0}, \quad (4.94)$$

*then* $\int_X G_3^p(t)$ *and both*

$$\bigg(\int_X\|D^3\mathrm{Rm}(t)\|^{2p}\bigg)^{\frac{1}{2p}} \text{ and } \bigg(\int_X\|D^4T(t)\|^{2p}\bigg)^{\frac{1}{2p}} \quad (4.95)$$

*are uniformly bounded along the Anomaly flow.*

**Corollary 4.3.2.** *Suppose that the assumptions* (4.14) - (4.16) *hold for* $k = 2$. *If*

$$\alpha' < \frac{1}{26a_0 B^2 C_0}, \tag{4.96}$$

*then there exist positive constants* $C_q$ *for* $q \geq 2$ *such that*

$$\|D^q \mathrm{Rm}\|, \|D^{q+1}T\|, \|D^{q+1}\overline{T}\| \leq C_q \tag{4.97}$$

*along the Anomaly flow on* $t \in [0, \tau)$.

As before, we see that these constants $C_q$ for $q \geq 2$ will depend on $g_0$, $\alpha'$, $B$, $C_0$, and $C_1$.

### 4.3.2 The $k = 1$ Case

We now aim to show the case when $k = 1$. As previously mentioned, this case is more complex as the estimates on $\|D^k \mathrm{Rm}\|$ and $\|D^{k+1}T\|$ also need to be reproven.

**Estimates on $\|D\mathrm{Rm}\|$ and $\|D^2 T\|$**

As before, we can check that

$$\begin{aligned}
\frac{\partial}{\partial t}\Big(\|D\mathrm{Rm}\|^2 + \|D^2T\|^2\Big) &\leq \frac{1}{2} \cdot \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot \Delta_R\Big(\|D\mathrm{Rm}\|^2 + \|D^2T\|^2\Big) \\
&- B^{-1}p \cdot \Big(\|D^2\mathrm{Rm}\|^2 + \|D^3T\|^2\Big) \\
&+ \sum_{m+l=1} 2\alpha' \mathrm{Re}\Big(\overline{\nabla}_{\bar{j}} \Big\langle \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot \big(\nabla^{m+1}\overline{\nabla}^l(\mathrm{Rm} * \mathrm{Rm})\big), \nabla^m \overline{\nabla}^l \mathrm{Rm} \Big\rangle^{\bar{j}}\Big) \\
&+ \sum_{m'+l'=2} 2\alpha' \mathrm{Re}\Big(\nabla_i \Big\langle \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot \big(\nabla^{m'}\overline{\nabla}^{l'}(\mathrm{Rm} * \mathrm{Rm})\big), \nabla^{m'} \overline{\nabla}^{l'} T \Big\rangle^{i}\Big) \\
&+ C\epsilon^{-1} \cdot \Big(1 + \big(\|D\mathrm{Rm}\|^2 + \|D^2T\|^2\big)\Big) + \Big[C\epsilon + 10a_0 B\alpha'\Big] \cdot \|D\mathrm{Rm}\|^4 \\
&+ \Big[C\epsilon + 6a_0 BC_0\alpha' + 4a_0 B\alpha'\Big] \cdot \Big(\|D^2\mathrm{Rm}\|^2 + \|D^3T\|^2\Big).
\end{aligned} \tag{4.98}$$

As expected, the quartic term $\|D\mathrm{Rm}\|^4$ appears, and so we cannot use the same test function as before. To compensate for this term, we use that

$$\frac{\partial}{\partial t}\Big(\|\mathrm{Rm}\|^2 + \|DT\|^2\Big) \leq \frac{1}{2} \cdot \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot \Delta_R\Big(\|\mathrm{Rm}\|^2 + \|DT\|^2\Big)$$

$$
\begin{aligned}
&- B^{-1}p \cdot \Big(\|D\mathrm{Rm}\|^2 + \|D^2T\|^2\Big) \\
&+ 2\alpha'\mathrm{Re}\left(\overline{\nabla}_{\bar{j}}\Big\langle\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot\big(\nabla(\mathrm{Rm} * \mathrm{Rm})\big), \mathrm{Rm}\Big\rangle^{\bar{j}}\right) \\
&+ \sum_{m'+l'=1} 2\alpha'\mathrm{Re}\left(\nabla_i\Big\langle\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot\big(\nabla^{m'}\overline{\nabla}^{l'}(\mathrm{Rm} * \mathrm{Rm})\big), \nabla^{m'}\overline{\nabla}^{l'}T\Big\rangle^{i}\right) \\
&+ C\epsilon^{-1}\cdot 1 + \Big[C\epsilon + 6a_0BC_0\alpha'\Big]\cdot\Big(\|D\mathrm{Rm}\|^2 + \|D^2T\|^2\Big)
\end{aligned}
\tag{4.99}
$$

and incorporate it into our test function.

Let $\mu > 0$ be a constant to be determined later. We consider a test function of the form

$$
\begin{aligned}
G &= \Big[\alpha'\cdot\Big(\|\mathrm{Rm}\|^2 + \|DT\|^2\Big) + \mu\Big]\cdot\Big(\|D\mathrm{Rm}\|^2 + \|D^2T\|^2\Big) \\
&= (\alpha'\cdot G_0 + \mu)\cdot G_1.
\end{aligned}
\tag{4.100}
$$

Using the two previous calculations, we have

$$
\begin{aligned}
\frac{\partial}{\partial t}G &= \alpha'\cdot\frac{\partial}{\partial t}\Big(\|\mathrm{Rm}\|^2 + \|DT\|^2\Big)\cdot\Big(\|D\mathrm{Rm}\|^2 + \|D^2T\|^2\Big) \\
&\quad + \Big[\alpha'\cdot\Big(\|\mathrm{Rm}\|^2 + \|DT\|^2\Big) + \mu\Big]\cdot\frac{\partial}{\partial t}\Big(\|D\mathrm{Rm}\|^2 + \|D^2T\|^2\Big) \\
&= \frac{1}{2}\cdot\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot\alpha'\cdot\Big(\|D\mathrm{Rm}\|^2 + \|D^2T\|^2\Big)\cdot\Delta_R\Big(\|\mathrm{Rm}\|^2 + \|DT\|^2\Big) \\
&\quad + \frac{1}{2}\cdot\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot\Big[\alpha'\cdot\Big(\|\mathrm{Rm}\|^2 + \|DT\|^2\Big) + \mu\Big] \\
&\qquad \cdot\Delta_R\Big(\|D\mathrm{Rm}\|^2 + \|D^2T\|^2\Big) \\
&\quad - B^{-1}\alpha'\cdot\Big(\|D\mathrm{Rm}\|^2 + \|D^2T\|^2\Big) \\
&\quad - B^{-1}\cdot\Big[\alpha'\cdot\Big(\|\mathrm{Rm}\|^2 + \|DT\|^2\Big) + \mu\Big]\cdot\Big(\|D^2\mathrm{Rm}\|^2 + \|D^3T\|^2\Big) \\
&\quad + 2\mathrm{Re}\,(\mathbf{A}) + 2\mathrm{Re}\,(\mathbf{B}) \\
&\quad + C\epsilon^{-1}\alpha'\cdot\Big(\|D\mathrm{Rm}\|^2 + \|D^2T\|^2\Big) \\
&\quad + C\epsilon^{-1}\cdot\Big[\alpha'\cdot\Big(\|\mathrm{Rm}\|^2 + \|DT\|^2\Big) + \mu\Big]\cdot\Big(1 + \|D\mathrm{Rm}\|^2 + \|D^2T\|^2\Big) \\
&\quad + \Big[C\epsilon + 6a_0BC_0\alpha'\Big]\cdot\alpha'\cdot\Big(\|D\mathrm{Rm}\|^2 + \|D^2T\|^2\Big)^2 \\
&\quad + \Big[C\epsilon + 10a_0B\alpha'\Big]\cdot\Big[\alpha'\cdot\Big(\|\mathrm{Rm}\|^2 + \|DT\|^2\Big) + \mu\Big] \\
&\qquad \cdot\Big(\|D\mathrm{Rm}\|^2 + \|D^2T\|^2\Big)^2
\end{aligned}
$$

$$+ \Big[C\epsilon + 6a_0BC_0\alpha' + 4a_0B\alpha'\Big] \cdot \Big[\alpha' \cdot \Big(\|\mathrm{Rm}\|^2 + \|DT\|^2\Big) + \mu\Big]$$
$$\cdot \Big(\|D^2\mathrm{Rm}\|^2 + \|D^3T\|^2\Big), \tag{4.101}$$

where we have used that $\|D\mathrm{Rm}\|^4 \leq \Big(\|D\mathrm{Rm}\|^2 + \|D^2T\|^2\Big)^2$ as necessary. In the above, the terms **(A)** and **(B)** are given by

$$\begin{aligned}(\mathbf{A}) = (\alpha')^2 \cdot \Big(\|D\mathrm{Rm}\|^2 + \|D^2T\|^2\Big) \\ \cdot \Big(\overline{\nabla}_{\bar{j}}\Big\langle\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot \big(\nabla(\mathrm{Rm} * \mathrm{Rm})\big), \mathrm{Rm}\Big\rangle^{\bar{j}}\Big) \\ + \sum_{m'+l'=1} (\alpha')^2 \cdot \Big(\|D\mathrm{Rm}\|^2 + \|D^2T\|^2\Big) \\ \cdot \Big(\nabla_i\Big\langle\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot \big(\nabla^{m'}\overline{\nabla}^{l'}(\mathrm{Rm} * \mathrm{Rm})\big), \nabla^{m'}\overline{\nabla}^{l'}T\Big\rangle^i\Big),\end{aligned} \tag{4.102}$$

$$\begin{aligned}(\mathbf{B}) = \sum_{m+l=1} \alpha' \cdot \Big[\alpha' \cdot \Big(\|\mathrm{Rm}\|^2 + \|DT\|^2\Big) + \mu\Big] \\ \cdot \Big(\overline{\nabla}_{\bar{j}}\Big\langle\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot \big(\nabla^{m+1}\overline{\nabla}^{l}(\mathrm{Rm} * \mathrm{Rm})\big), \nabla^m\overline{\nabla}^l\mathrm{Rm}\Big\rangle^{\bar{j}}\Big) \\ + \sum_{m'+l'=2} \alpha' \cdot \Big[\alpha' \cdot \Big(\|\mathrm{Rm}\|^2 + \|DT\|^2\Big) + \mu\Big] \\ \cdot \Big(\nabla_i\Big\langle\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot \big(\nabla^{m'}\overline{\nabla}^{l'}(\mathrm{Rm} * \mathrm{Rm})\big), \nabla^{m'}\overline{\nabla}^{l'}T\Big\rangle^i\Big).\end{aligned} \tag{4.103}$$

By further grouping terms and inflating constants, while noting that $\alpha'\Big(\|\mathrm{Rm}\|^2 + \|DT\|^2\Big) + \mu \leq 2C_0^2 + \mu$, we get

$$\begin{aligned}\frac{\partial}{\partial t}G \leq \frac{1}{2} \cdot \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot (\Delta_R G) \\ - B^{-1}\alpha'\Big(\|D\mathrm{Rm}\|^2 + \|D^2T\|^2\Big)^2 - B^{-1}\mu \cdot \Big(\|D^2\mathrm{Rm}\|^2 + \|D^3T\|^2\Big) \\ + \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot 2\alpha'\mathrm{Re}\Big\langle\nabla\Big(\|\mathrm{Rm}\|^2 + \|DT\|^2\Big), \nabla\Big(\|D\mathrm{Rm}\|^2 + \|D^2T\|^2\Big\rangle \\ + 2\mathrm{Re}\,(\mathbf{A}) + 2\mathrm{Re}\,(\mathbf{B}) + C\epsilon^{-1} \cdot \Big(1 + G\Big)\end{aligned}$$

$$+ \Big[C\epsilon + 20a_0BC_0^2(\alpha')^2 + 6a_0BC_0(\alpha')^2 + 10a_0B\alpha'\mu\Big]$$
$$\cdot \Big(\|D\text{Rm}\|^2 + \|D^2T\|^2\Big)^2$$
$$+ \Big[C\epsilon + 12a_0BC_0^3(\alpha')^2 + 8a_0BC_0^2(\alpha')^2 + 6a_0BC_0\alpha'\mu + 4a_0B\alpha'\mu\Big]$$
$$\cdot \Big(\|D^2\text{Rm}\|^2 + \|D^3T\|^2\Big). \tag{4.104}$$

Note here that the generic constant $C$ may depend on $\mu$ as well.

From this, it follows that for $p \geq 3$, that

$$\frac{\partial}{\partial t}\left(\int_X G^p\right) \leq \frac{1}{2}\cdot\int_X \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot(\Delta_R G^p) - B^{-1}p(p-1)\int_X G^{p-2}\cdot\|\nabla G\|^2$$
$$- B^{-1}\alpha' p\int_X G^{p-1}\cdot\Big(\|D\text{Rm}\|^2 + \|D^2T\|^2\Big)^2$$
$$- B^{-1}\mu p\int_X G^{p-1}\cdot\Big(\|D^2\text{Rm}\|^2 + \|D^3T\|^2\Big)$$
$$+ 2p\alpha'\text{Re}\left(\int_X G^{p-1}\cdot\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\right.$$
$$\left.\cdot\Big\langle\nabla\Big(\|\text{Rm}\|^2 + \|DT\|^2\Big), \nabla\Big(\|D\text{Rm}\|^2 + \|D^2T\|^2\Big)\Big\rangle\right)$$
$$+ 2p\text{Re}\left(\int_X G^{p-1}\cdot(\mathbf{A})\right) + 2p\text{Re}\left(\int_X G^{p-1}\cdot(\mathbf{B})\right)$$
$$+ C\epsilon^{-1}\cdot\int_X G^{p-1}\cdot\Big(1+G\Big)$$
$$+ \Big[C\epsilon + 20a_0BC_0^2(\alpha')^2p + 6a_0BC_0(\alpha')^2p + 10a_0B\alpha'\mu p\Big]$$
$$\cdot\int_X G^{p-1}\cdot\Big(\|D\text{Rm}\|^2 + \|D^2T\|\Big)^2$$
$$+ \Big[C\epsilon + 12a_0BC_0^3(\alpha')^2p + 8a_0BC_0^2(\alpha')^2p + 6a_0BC_0\alpha'\mu p + 4a_0B\alpha'\mu p\Big]$$
$$\cdot\int_X G^{p-1}\cdot\Big(\|D^2\text{Rm}\|^2 + \|D^3T\|^2\Big). \tag{4.105}$$

As seen in previous sections, the Laplacian term can be absorbed into the other terms

$$\frac{1}{2} \cdot \int_X \Big( \frac{1}{2\|\Upsilon\|_\omega} \Big) \cdot (\Delta_R G^p) \leq C\epsilon^{-1} \cdot \int_X G^p + C\epsilon \cdot \int_X G^{p-2} \cdot \|\nabla G\|^2. \tag{4.106}$$

In preparation for dealing with the upcoming terms, we note that by the (Ricci) Commutator Identity (A.18), we have

$$\begin{aligned} \big\|\nabla \|D\mathrm{Rm}\|^2\big\| &\leq \big\|\nabla\langle\nabla\mathrm{Rm}, \nabla\mathrm{Rm}\rangle\big\| + \big\|\nabla\langle\overline{\nabla}\mathrm{Rm}, \overline{\nabla}\mathrm{Rm}\rangle\big\| \\ &\leq \|\nabla\mathrm{Rm}\| \cdot \|\nabla^2\mathrm{Rm}\| + |\nabla\mathrm{Rm}\|\|\overline{\nabla}\nabla\mathrm{Rm}\| \\ &\quad + \|\overline{\nabla}\mathrm{Rm}\| \cdot \|\nabla\overline{\nabla}\mathrm{Rm}\| + \|\overline{\nabla}\mathrm{Rm}\| \cdot \|\overline{\nabla}^2\mathrm{Rm}\| \\ &\leq C \cdot \|\mathrm{Rm}\| \cdot \|D\mathrm{Rm}\| + 4\|D\mathrm{Rm}\| \cdot \|D^2\mathrm{Rm}\|. \end{aligned} \tag{4.107}$$

Likewise, we have

$$\begin{aligned} \big\|\nabla\|D^2T\|^2\big\| &\leq \big\|\nabla\langle\nabla^2 T, \nabla^2 T\rangle\big\| + \big\|\nabla\langle\nabla\overline{\nabla}T, \nabla\overline{\nabla}T\rangle\big\| + \|\nabla\langle\overline{\nabla}^2 T, \overline{\nabla}^2 T\rangle\| \\ &\leq C \cdot \|DT\| \cdot \|D^2T\| + C \cdot \|D\mathrm{Rm}\| \cdot \|D^2T\| \\ &\quad + 6\|D^2T\| \cdot \|D^3T\|, \end{aligned} \tag{4.108}$$

$$\big\|\nabla\|\mathrm{Rm}\|^2\big\| \leq 2\|\mathrm{Rm}\| \cdot \|D\mathrm{Rm}\|, \tag{4.109}$$

$$\big\|\nabla\|DT\|^2\big\| \leq C \cdot \|\mathrm{Rm}\| \cdot \|DT\| + 4\|DT\| \cdot \|D^2T\|. \tag{4.110}$$

As before, we keep track of the coefficients on the highest-order terms.

The inner product term following the negative terms in (4.105) is bounded by

$$\begin{aligned} &2p\alpha'\mathrm{Re}\Bigg( \int_X G^{p-1} \cdot \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \\ &\qquad \cdot \Big\langle \nabla\Big(\|\mathrm{Rm}\|^2 + \|DT\|^2\Big), \nabla\Big(\|D\mathrm{Rm}\|^2 + \|D^2T\|^2\Big)\Big\rangle\Bigg) \\ &\leq 2B\alpha' p \cdot \int_X G^{p-1} \cdot \Big(C \cdot \|\mathrm{Rm}\| \cdot \|DT\| \\ &\qquad + 2\|\mathrm{Rm}\| \cdot \|D\mathrm{Rm}\| + 4\|DT\| \cdot \|D^2T\|\Big) \\ &\qquad \cdot \Big(C \cdot \|\mathrm{Rm}\| \cdot \|D\mathrm{Rm}\| + C \cdot \|DT\| \cdot \|D^2T\| + C \cdot \|D\mathrm{Rm}\| \cdot \|D^2T\| \\ &\qquad + 4\|D\mathrm{Rm}\| \cdot \|D^2\mathrm{Rm}\| + 6\|D^2T\| \cdot \|D^3T\|\Big). \end{aligned} \tag{4.111}$$

In the above, those terms not involving the generic constant $C$ are the ones that contribute to the highest-order terms. We apply Young's Inequality to the other terms to get

$$
\begin{aligned}
&2\alpha' p \mathrm{Re}\Bigg( \int_X G^{p-1} \cdot \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \\
&\qquad \cdot \Big\langle \nabla\Big(\|\mathrm{Rm}\|^2 + \|DT\|^2\Big), \nabla\Big(\|D\mathrm{Rm}\|^2 + \|D^2T\|^2\Big)\Big\rangle \Bigg) \\
&\le C\epsilon^{-1} \cdot \int_X \Big(G^{p-1} + G^p\Big) + C\epsilon \cdot \int_X G^{p-1} \cdot \Big(\|D\mathrm{Rm}\|^2 + \|D^2T\|^2\Big)^2 \\
&\quad + C\epsilon \cdot \int_X G^{p-1} \cdot \Big(\|D^2\mathrm{Rm}\|^2 + \|D^3T\|^2\Big) \\
&\quad + 16B\alpha' p \cdot \int_X G^{p-1} \cdot \|\mathrm{Rm}\| \cdot \|D\mathrm{Rm}\|^2 \cdot \|D^2\mathrm{Rm}\| \\
&\quad + 24B\alpha' p \cdot \int_X G^{p-1} \cdot \|\mathrm{Rm}\| \cdot \|D\mathrm{Rm}\| \cdot \|D^2T\| \cdot \|D^3T\| \\
&\quad + 32B\alpha' p \cdot \int_X G^{p-1} \cdot \|DT\| \cdot \|D\mathrm{Rm}\| \cdot \|D^2T\| \cdot \|D^2\mathrm{Rm}\| \\
&\quad + 48B\alpha' p \cdot \int_X G^{p-1} \cdot \|DT\| \cdot \|D^2T\|^2 \cdot \|D^3T\| \\
&\le C\epsilon^{-1} \cdot \int_X \Big(G^{p-1} + G^p\Big) + C\epsilon \cdot \int_X G^{p-1} \cdot \Big(\|D\mathrm{Rm}\|^2 + \|D^2T\|^2\Big)^2 \\
&\quad + C\epsilon \cdot \int_X G^{p-1} \cdot \Big(\|D^2\mathrm{Rm}\|^2 + \|D^3T\|^2\Big) \\
&\quad + \frac{1}{8}B^{-1}\alpha' p \cdot \int_X G^{p-1} \cdot \|D\mathrm{Rm}\|^4 \\
&\qquad + 512B^3\alpha' p \cdot \int_X G^{p-1} \cdot \|\mathrm{Rm}\|^2 \cdot \|D^2\mathrm{Rm}\|^2 \\
&\quad + \frac{1}{8}B^{-1}\alpha' p \cdot \int_X G^{p-1} \cdot \|D\mathrm{Rm}\|^2 \cdot \|D^2T\|^2 \\
&\qquad + 1152B^3\alpha' p \cdot \int_X G^{p-1} \cdot \|\mathrm{Rm}\|^2 \cdot \|D^3\mathrm{Rm}\|^2 \\
&\quad + \frac{1}{8}B^{-1}\alpha' p \cdot \int_X G^{p-1} \cdot \|D\mathrm{Rm}\|^2 \cdot \|D^2T\|^2 \\
&\qquad + 2048B^3\alpha' p \cdot \int_X G^{p-1} \cdot \|DT\|^2 \cdot \|D^2\mathrm{Rm}\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{8}B^{-1}\alpha' p \cdot \int_X G^{p-1} \cdot \|D^2 T\|^4 \\
&\quad + 4608 B^3 \alpha' p \cdot \int_X G^{p-1} \cdot \|DT\|^2 \cdot \|D^3 \mathrm{Rm}\|^2 \\
&\leq C\epsilon^{-1} \cdot \int_X \Big( G^{p-1} + G^p \Big) \\
&\quad + \Big[ C\epsilon + \frac{1}{2} B^{-1} \alpha' p \Big] \cdot \int_X G^{p-1} \cdot \Big( \|D\mathrm{Rm}\|^2 + \|D^2 T\|^2 \Big)^2 \\
&\quad + \Big[ C\epsilon + 8320 B^3 C_0^2 \alpha' p \Big] \cdot \int_X G^{p-1} \cdot \Big( \|D^2\mathrm{Rm}\|^2 + \|D^3 T\|^2 \Big). \qquad (4.112)
\end{aligned}
$$

It remains to deal with the terms involving **(A)** and **(B)**. For part of the **(A)** term, we have

$$
\begin{aligned}
&2(\alpha')^2 p \mathrm{Re} \Bigg( \int_X G^{p-1} \cdot \Big( \|D\mathrm{Rm}\|^2 + \|D^2 T\|^2 \Big) \\
&\qquad \cdot \Big( \overline{\nabla}_{\bar{j}} \Big\langle \Big( \frac{1}{2\|\Upsilon\|_\omega} \Big) \cdot \big( \nabla(\mathrm{Rm} * \mathrm{Rm}) \big), \mathrm{Rm} \Big\rangle^{\bar{j}} \Big) \Bigg) \\
&= 2(\alpha')^2 p \mathrm{Re} \Bigg( \int_X \overline{\nabla}_{\bar{j}} \Big\langle G^{p-1} \cdot \Big( \frac{1}{2\|\Upsilon\|_\omega} \Big) \cdot \Big( \|D\mathrm{Rm}\|^2 + \|D^2 T\|^2 \Big) \\
&\qquad \cdot \big( \nabla(\mathrm{Rm} * \mathrm{Rm}) \big), \mathrm{Rm} \Big\rangle^{\bar{j}} \Bigg) \\
&\quad - 2(\alpha')^2 p \mathrm{Re} \Bigg( \int_X \overline{\nabla} G^{p-1} \\
&\qquad \cdot \Big\langle \Big( \frac{1}{2\|\Upsilon\|_\omega} \Big) \cdot \Big( \|D\mathrm{Rm}\|^2 + \|D^2 T\|^2 \Big) \cdot \big( \nabla(\mathrm{Rm} * \mathrm{Rm}) \big), \mathrm{Rm} \Big\rangle \Bigg) \\
&\quad - 2(\alpha')^2 p \mathrm{Re} \Bigg( \int_X G^{p-1} \cdot \Big\langle \Big( \frac{1}{2\|\Upsilon\|_\omega} \Big) \cdot \big( \nabla(\mathrm{Rm} * \mathrm{Rm}) \big), \mathrm{Rm} \Big\rangle \\
&\qquad \cdot \overline{\nabla} \Big( \|D\mathrm{Rm}\|^2 + \|D^2 T\|^2 \Big) \Bigg)
\end{aligned}
$$

$$\begin{aligned}
&\leq C \cdot \int_X G^{p-1} \cdot \Big( \|D\mathrm{Rm}\|^2 + \|D^2 T\|^2 \Big) \cdot \|D\mathrm{Rm}\| \\
&\quad + 4a_0 B C_0^2 (\alpha')^2 p(p-1) \\
&\qquad \cdot \int_X G^{p-2} \cdot \|\overline{\nabla} G\| \cdot \Big( \|D\mathrm{Rm}\|^2 + \|D^2 T\|^2 \Big) \cdot \|D\mathrm{Rm}\| \\
&\quad + 4a_0 B C_0^2 (\alpha')^2 p \\
&\qquad \cdot \int_X G^{p-1} \cdot \|D\mathrm{Rm}\| \cdot \Big( C \cdot \|\mathrm{Rm}\| \cdot \|D\mathrm{Rm}\| + C \cdot \|DT\| \cdot \|D^2 T\| \\
&\qquad\qquad + C \cdot \|D\mathrm{Rm}\| \cdot \|D^2 T\| + 4\|D\mathrm{Rm}\| \cdot \|D^2 \mathrm{Rm}\| \\
&\qquad\qquad + 6\|D^2 T\| \cdot \|D^3 T\| \Big). \qquad (4.113)
\end{aligned}$$

We apply Young's Inequality to the terms in each integral and keep track of the coefficients on the highest-order contributions in the third integral, while absorbing the rest into "good" terms with the generic constant $C$.

$$\begin{aligned}
&2(\alpha')^2 p \mathrm{Re} \Bigg( \int_X G^{p-1} \cdot \Big( \|D\mathrm{Rm}\|^2 + \|D^2 T\|^2 \Big) \\
&\qquad \cdot \Big( \overline{\nabla}_{\bar{j}} \Big\langle \Big( \frac{1}{2\|\Upsilon\|_\omega} \Big) \cdot (\nabla(\mathrm{Rm} * \mathrm{Rm})), \mathrm{Rm} \Big\rangle^{\bar{j}} \Big) \Bigg) \\
&\leq C\epsilon^{-1} \cdot \int_X G^p + C\epsilon \cdot \int_X G^{p-1} \cdot \Big( \|D\mathrm{Rm}\|^2 + \|D^2 T\|^2 \Big)^2 \\
&\quad + 2a_0 B C_0^2 (\alpha')^2 p(p-1) \cdot \int_X G^{p-3} \cdot \Big( \|D\mathrm{Rm}\|^2 + \|D^2 T\|^2 \Big) \cdot \|\overline{\nabla} G\|^2 \\
&\quad + 2a_0 B C_0^2 (\alpha')^2 p(p-1) \cdot \int_X G^{p-3} \cdot G^2 \cdot \|D\mathrm{Rm}\|^2 \cdot \Big( \|D\mathrm{Rm}\|^2 + \|D^2 T\|^2 \Big) \\
&\quad + 20a_0 B C_0^2 (\alpha') p \cdot \int_X G^{p-1} \cdot \Big( \|D\mathrm{Rm}\|^2 + \|D^2 T\|^2 \Big)^2 \\
&\quad + 20a_0 B C_0^2 (\alpha') p \cdot \int_X G^{p-1} \cdot \Big( \|D^2\mathrm{Rm}\|^2 + \|D^3 T\|^2 \Big) \\
&\leq C\epsilon^{-1} \cdot \int_X G^p + C\epsilon \cdot \int_X G^{p-1} \cdot \Big( \|D\mathrm{Rm}\|^2 + \|D^2 T\|^2 \Big)^2 \\
&\quad + 2a_0 B C_0^2 (\alpha')^2 \mu^{-1} p(p-1) \cdot \int_X G^{p-2} \cdot \|\overline{\nabla} G\|^2 \\
&\quad + 2a_0 B C_0^2 (\alpha')^2 p(p-1) \cdot \int_X G^{p-1} \cdot \Big( \|D\mathrm{Rm}\|^2 + \|D^2 T\|^2 \Big)^2
\end{aligned}$$

$$
\begin{aligned}
&+ 20a_0BC_0^2(\alpha')p \cdot \int_X G^{p-1} \cdot \Big(\|D\mathrm{Rm}\|^2 + \|D^2T\|^2\Big)^2 \\
&+ 20a_0BC_0^2(\alpha')p \cdot \int_X G^{p-1} \cdot \Big(\|D^2\mathrm{Rm}\|^2 + \|D^3T\|^2\Big) \\
\leq\ & C\epsilon^{-1} \cdot \int_X G^p + 2a_0BC_0^2(\alpha')^2\mu^{-1}p(p-1) \cdot \int_X G^{p-2} \cdot \|\overline{\nabla}G\|^2 \\
&+ \Big[C\epsilon + 2a_0BC_0^2(\alpha')^2p(p-1) + 20a_0BC_0^2(\alpha')p\Big] \\
&\quad \cdot \int_X G^{p-1} \cdot \Big(\|D\mathrm{Rm}\|^2 + \|D^2T\|^2\Big)^2 \\
&+ 20a_0BC_0^2(\alpha')^2p \cdot \int_X G^{p-1} \cdot \Big(\|D^2\mathrm{Rm}\|^2 + \|D^3T\|^2\Big).
\end{aligned} \tag{4.114}
$$

The other parts of the **(A)** term have similar bounds and so

$$
\begin{aligned}
&2p\mathrm{Re}\left(\int_X G^{p-1} \cdot (\mathbf{A})\right) \\
&\leq C\epsilon^{-1} \cdot \int_X G^p + 6a_0BC_0^2(\alpha')^2\mu^{-1}p(p-1) \cdot \int_X G^{p-2} \cdot \|\overline{\nabla}G\|^2 \\
&\quad + \Big[C\epsilon + 6a_0BC_0^2(\alpha')^2p(p-1) + 60a_0BC_0^2(\alpha')p\Big] \\
&\quad\quad \cdot \int_X G^{p-1} \cdot \Big(\|D\mathrm{Rm}\|^2 + \|D^2T\|^2\Big)^2 \\
&\quad + 60a_0BC_0^2(\alpha')^2p \cdot \int_X G^{p-1} \cdot \Big(\|D^2\mathrm{Rm}\|^2 + \|D^3T\|^2\Big).
\end{aligned} \tag{4.115}
$$

For the **(B)** term, we check that

$$
\begin{aligned}
&2\alpha' p\mathrm{Re}\Bigg(\int_X G^{p-1} \cdot \Big[\alpha' \cdot \Big(\|\mathrm{Rm}\|^2 + \|DT\|^2\Big) + \mu\Big] \\
&\quad \cdot \overline{\nabla}_{\bar{j}}\Big\langle \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot \big(\nabla^2(\mathrm{Rm} * \mathrm{Rm})\big), \nabla\mathrm{Rm}\Big\rangle^{\bar{j}}\Bigg) \\
&= 2\alpha' p\mathrm{Re}\Bigg(\int_X \overline{\nabla}_{\bar{j}}\Big\langle G^{p-1} \cdot \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \\
&\quad\quad \cdot \Big[\alpha' \cdot \Big(\|\mathrm{Rm}\|^2 + \|DT\|^2\Big) + \mu\Big] \cdot \big(\nabla^2(\mathrm{Rm} * \mathrm{Rm})\big), \nabla\mathrm{Rm}\Big\rangle^{\bar{j}}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&- 2\alpha' p \mathrm{Re}\Bigg( \int_X \overline{\nabla} G^{p-1} \cdot \Big( \frac{1}{2\|\Upsilon\|_\omega} \Big) \\
&\qquad \cdot \Big\langle \Big[ \alpha' \cdot \Big( \|\mathrm{Rm}\|^2 + \|DT\|^2 \Big) + \mu \Big] \cdot \big( \nabla(\mathrm{Rm} * \mathrm{Rm}) \big), \nabla \mathrm{Rm} \Big\rangle \Bigg) \\
&- 2(\alpha')^2 p \mathrm{Re}\Bigg( \int_X G^{p-1} \cdot \Big( \frac{1}{2\|\Upsilon\|_\omega} \Big) \\
&\qquad \cdot \Big\langle \big( \nabla^2(\mathrm{Rm} * \mathrm{Rm}) \big), \nabla \mathrm{Rm} \Big\rangle \cdot \overline{\nabla} \Big( \|\mathrm{Rm}\|^2 + \|DT\|^2 \Big) \Bigg) \\
\leq\ & C \cdot \Bigg( \int_X G^{p-1} \cdot \Big[ \alpha' \cdot \Big( \|\mathrm{Rm}\|^2 + \|DT\|^2 \Big) + \mu \Big] \\
&\qquad \cdot \Big( \|\mathrm{Rm}\| \cdot \|D^2 \mathrm{Rm}\| + \|D\mathrm{Rm}\|^2 \Big) \cdot \|D\mathrm{Rm}\| \Bigg) \\
&+ 2a_0 B \alpha' p(p-1) \cdot \Bigg( \int_X G^{p-2} \cdot \|\overline{\nabla} G\| \cdot \Big[ \alpha' \cdot \Big( \|\mathrm{Rm}\|^2 + \|DT\|^2 \Big) + \mu \Big] \\
&\qquad \cdot \Big( 2\|\mathrm{Rm}\| \cdot \|D^2 \mathrm{Rm}\| + 2\|D\mathrm{Rm}\|^2 \Big) \cdot \|D\mathrm{Rm}\| \Bigg) \\
&+ 2a_0 B (\alpha')^2 p \cdot \Bigg( \int_X G^{p-1} \cdot \Big( 2\|\mathrm{Rm}\| \cdot \|D^2 \mathrm{Rm}\| + 2\|D\mathrm{Rm}\|^2 \Big) \cdot \|D\mathrm{Rm}\| \\
&\qquad \cdot \Big( C \cdot \|\mathrm{Rm}\| \cdot \|DT\| + 2\|\mathrm{Rm}\| \cdot \|D\mathrm{Rm}\| + 4\|DT\| \cdot \|D^2 T\| \Big) \Bigg).
\end{aligned}
\tag{4.116}
$$

A similar computation to the **(A)** term then gives that

$$
\begin{aligned}
&2\alpha' p \mathrm{Re}\Bigg( \int_X G^{p-1} \cdot \Big[ \alpha' \cdot \Big( \|\mathrm{Rm}\|^2 + \|DT\|^2 \Big) + \mu \Big] \\
&\qquad \cdot \overline{\nabla}_{\bar{j}} \Big\langle \Big( \frac{1}{2\|\Upsilon\|_\omega} \Big) \cdot \big( \nabla^2(\mathrm{Rm} * \mathrm{Rm}) \big), \nabla \mathrm{Rm} \Big\rangle^{\bar{j}} \Bigg) \\
&\leq C\epsilon^{-1} \cdot \int_X \Big( G^{p-1} + G^p \Big) \\
&\quad + C\epsilon \cdot \int_X G^{p-1} \cdot \Big( \|D\mathrm{Rm}\|^2 + \|D^2 T\|^2 \Big)^2
\end{aligned}
$$

$$
\begin{aligned}
&+ C\epsilon \cdot \int_X G^{p-1} \cdot \Big( \|D^2\mathrm{Rm}\|^2 + \|D^3 T\|^2 \Big) \\
&+ 2a_0 B\alpha' p(p-1) \\
&\quad \cdot \int_X G^{p-3} \cdot \Big[ \alpha' \cdot \Big( \|\mathrm{Rm}\|^2 + \|DT\|^2 \Big) + \mu \Big] \cdot \|D\mathrm{Rm}\|^2 \cdot \|\overline{\nabla} G\|^2 \\
&+ 2a_0 B\alpha' p(p-1) \\
&\quad \cdot \int_X G^{p-1} \cdot \Big[ \alpha' \cdot \Big( \|\mathrm{Rm}\|^2 + \|DT\|^2 \Big) + \mu \Big] \cdot \Big( \|D\mathrm{Rm}\|^2 + \|D^2 T\|^2 \Big)^2 \\
&+ 2a_0 B C_0^2 \alpha' p(p-1) \\
&\quad \cdot \int_X G^{p-1} \cdot \Big[ \alpha' \cdot \Big( \|\mathrm{Rm}\|^2 + \|DT\|^2 \Big) + \mu \Big] \cdot \Big( \|D^2\mathrm{Rm}\|^2 + \|D^3 T\|^2 \Big) \\
&+ 12 a_0 B C_0^2 (\alpha')^2 p \cdot \int_X G^{p-1} \cdot \Big( \|D\mathrm{Rm}\|^2 + \|D^2 T\|^2 \Big)^2 \\
&+ 12 a_0 B C_0^2 (\alpha')^2 p \cdot \int_X G^{p-1} \cdot \Big( \|D^2\mathrm{Rm}\|^2 + \|D^3 T\|^2 \Big) \\
&+ 16 a_0 B C_0 (\alpha')^2 p \cdot \int_X G^{p-1} \cdot \Big( \|D\mathrm{Rm}\|^2 + \|D^2 T\|^2 \Big)^2 \\
\le\ & C\epsilon^{-1} \cdot \int_X \Big( G^{p-1} + G^p \Big) + 2a_0 B\alpha' p(p-1) \cdot \int_X G^{p-2} \cdot \|\overline{\nabla} G\|^2 \\
&+ \Big[ C\epsilon + 4a_0 B C_0^2 (\alpha')^2 p(p-1) + 12 a_0 B C_0^2 (\alpha')^2 p + 16 a_0 B C_0^2 (\alpha')^2 p \\
&\quad + 2a_0 B\alpha' \mu p(p-1) \Big] \cdot \int_X G^{p-1} \cdot \Big( \|D\mathrm{Rm}\|^2 + \|D^2 T\|^2 \Big)^2 \\
&+ \Big[ C\epsilon + 4a_0 B C_0^4 (\alpha')^2 p(p-1) + 12 a_0 B C_0^2 (\alpha')^2 p \\
&\quad + 2a_0 B C_0^2 \alpha' \mu p(p-1) \Big] \cdot \int_X G^{p-1} \cdot \Big( \|D^2\mathrm{Rm}\|^2 + \|D^3 T\|^2 \Big),
\end{aligned} \tag{4.117}
$$

where we have again used that $\alpha' \cdot \Big( \|\mathrm{Rm}\|^2 + \|DT\|^2 \Big) + \mu \le 2C_0^2 + \mu$.

The other parts of the **(B)** term have similar bounds and so we have

$$\begin{aligned}
&2p\mathrm{Re}\left(\int_X G^{p-1}\cdot(\mathbf{B})\right)\\
&\le C\epsilon^{-1}\cdot\int_X\Big(G^{p-1}+G^p\Big)+10a_0B\alpha' p(p-1)\cdot\int_X G^{p-2}\cdot\|\overline{\nabla}G\|^2\\
&\quad+\Big[C\epsilon+20a_0BC_0^2(\alpha')^2p(p-1)+60a_0BC_0^2(\alpha')^2p+80a_0BC_0^2(\alpha')^2p\\
&\qquad+10a_0B\alpha'\mu p(p-1)\Big]\cdot\int_X G^{p-1}\cdot\Big(\|D\mathrm{Rm}\|^2+\|D^2T\|^2\Big)^2\\
&\quad+\Big[C\epsilon+20a_0BC_0^4(\alpha')^2p(p-1)+60a_0BC_0^2(\alpha')^2p\\
&\qquad+10a_0BC_0^2\alpha'\mu p(p-1)\Big]\cdot\int_X G^{p-1}\cdot\Big(\|D^2\mathrm{Rm}\|^2+\|D^3T\|^2\Big).
\end{aligned}\tag{4.118}$$

Combining everything together, we get

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\int_X G^p\right)\le C\epsilon^{-1}\cdot\int_X\Big(1+G^p\Big)\\
&\quad+\Big[C\epsilon+6a_0BC_0^2(\alpha')^2\mu^{-1}p(p-1)+10a_0B\alpha' p(p-1)\\
&\qquad-B^{-1}p(p-1)\Big]\cdot\int_X G^{p-2}\cdot\|\overline{\nabla}G\|^2\\
&\quad+\Big[C\epsilon+140a_0BC_0^2(\alpha')^2p+26a_0BC_0^2(\alpha')^2p(p-1)+86a_0BC_0(\alpha')^2p\\
&\qquad+10a_0B\alpha'\mu p^2-\frac{1}{2}B^{-1}\alpha' p\Big]\cdot\int_X G^{p-1}\cdot\Big(\|D\mathrm{Rm}\|^2+\|D^2T\|^2\Big)^2\\
&\quad+\Big[C\epsilon+20a_0BC_0^4(\alpha')^2p(p-1)+8320B^3C_0^2\alpha' p+12a_0BC_0^3(\alpha')^2p\\
&\qquad+128a_0BC_0^2(\alpha')^2p+10a_0BC_0^2\alpha'\mu p(p-1)+6a_0BC_0\alpha'\mu p\\
&\qquad+4a_0B\alpha'\mu p-B^{-1}\mu p\Big]\cdot\int_X G^{p-1}\cdot\Big(\|D^2\mathrm{Rm}\|^2+\|D^3T\|^2\Big).
\end{aligned}\tag{4.119}$$

Thus, reading off coefficients, we see that if

$$6a_0BC_0^2(\alpha')^2\mu^{-1}+10a_0B\alpha'<B^{-1},\tag{4.120}$$

$$140a_0BC_0^2\alpha'+26a_0BC_0^2\alpha'(p-1)+86a_0BC_0\alpha'+10a_0B\mu p<\frac{1}{2}B^{-1},\tag{4.121}$$

$$
\begin{aligned}
&20a_0BC_0^4(\alpha')^2(p-1) + 8320B^3C_0^2\alpha' \\
&\quad + 12a_0BC_0^3(\alpha')^2 + 128a_0BC_0^2(\alpha')^2 \\
&\quad + 10a_0BC_0^2\alpha'\mu(p-1) + 6a_0BC_0\alpha'\mu + 4a_0B\alpha'\mu < B^{-1}\mu, \qquad (4.122)
\end{aligned}
$$

then we can absorb the corresponding terms into the negative ones.

We see that if

$$
\alpha' = O\big(a_0^{-1}B^{-6}\max(1,C_0)^{-2}p^{-1}\big) \text{ and } \mu = O(a_0^{-1}B^{-2}p^{-1}), \qquad (4.123)
$$

then the three inequalities could hold.

To this end, we get that if

$$
\alpha' < \frac{1}{10^6 a_0 B^6 \max(1,C_0)^2 p}, \qquad (4.124)
$$

then by picking

$$
\mu = \frac{1}{100a_0B^2p}, \qquad (4.125)
$$

we can satisfy the inequalities (4.120) - (4.122) and can then choose $\epsilon = \epsilon(k,\alpha',p,\mu) > 0$ such that

$$
\frac{\partial}{\partial t}\left(\int_X G^p\right) \le C + C\cdot\int_X G^p. \qquad (4.126)
$$

As a consequence of Grönwall's Inequality again, we get

**Theorem 4.3.3.** *Let $k=1$ and set $G_q = \|D^q\mathrm{Rm}\|^2 + \|D^{q+1}T\|^2$. Suppose the assumptions* (4.14) - (4.16) *hold and set*

$$
\begin{aligned}
G &= \Big[\alpha'\cdot\Big(\|\mathrm{Rm}\|^2 + \|D^2T\|^2\Big) + \mu\Big]\cdot\Big(\|D\mathrm{Rm}\|^2 + \|D^2T\|^2\Big) \\
&= (\alpha'\cdot G_0 + \mu)\cdot G_1 \qquad (4.127)
\end{aligned}
$$

*for some $\mu > 0$.*

*i) If $p \ge 3$ and*

$$
\alpha' < \frac{1}{10^6 a_0 B^6 \max(1,C_0)^2 p}, \qquad (4.128)
$$

*then for*

$$
\mu = \frac{1}{100a_0B^2p} \qquad (4.129)
$$

*there exists some constant* $\Lambda_p = \Lambda_p(k, \alpha') > 0$ *such that*

$$\int_X G^p(t) \leq \left(1 + \int_X G^p(0)\right) e^{\Lambda_p t} < \left(1 + \int_X G^p(0)\right) e^{\Lambda_p \tau}. \tag{4.130}$$

*That is,* $\int_X G^p(t)$ *is uniformly bounded along the flow.*

*In particular, we get that* $\int_X G_1^p(t)$ *and both*

$$\left(\int_X \|D\mathrm{Rm}(t)\|^{2p}\right)^{\frac{1}{2p}} \text{ and } \left(\int_X \|D^2 T(t)\|^{2p}\right)^{\frac{1}{2p}} \tag{4.131}$$

*are bounded along the Anomaly flow;*

*ii) If instead* $p \in [1, 3)$ *and*

$$\alpha' < \frac{1}{3 \cdot 10^6 a_0 B^6 \max(1, C_0)^2}, \tag{4.132}$$

*then for*

$$\mu = \frac{1}{300 a_0 B^2}, \tag{4.133}$$

*the function* $\int_X G^p(t)$ *and also* $\int_X G_1^p(t)$ *and both*

$$\left(\int_X \|D\mathrm{Rm}(t)\|^{2p}\right)^{\frac{1}{2p}} \text{ and } \left(\int_X \|D^2 T(t)\|^{2p}\right)^{\frac{1}{2p}} \tag{4.134}$$

*are uniformly bounded along the Anomaly flow.*

**Estimates on $\|D^2\mathrm{Rm}\|$ and $\|D^3 T\|$**

For the higher-order estimates, one can check that

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big(\|D^2\mathrm{Rm}\|^2 + \|D^3 T\|^2\Big) \\
&\leq \frac{1}{2} \cdot \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot \Delta_R\Big(\|D^2\mathrm{Rm}\|^2 + \|D^3 T\|^2\Big) \\
&\quad - B^{-1} \cdot \Big(\|D^3\mathrm{Rm}\|^2 + \|D^4 T\|^2\Big) \\
&\quad + \sum_{m+l=2} 2\alpha' \mathrm{Re}\Big(\overline{\nabla}_{\bar{j}} \Big\langle \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot \Big(\nabla^{m+1}\overline{\nabla}^l (\mathrm{Rm} * \mathrm{Rm}), \nabla^m \overline{\nabla}^l \mathrm{Rm}\Big\rangle^{\bar{j}}\Big)
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{m'+l'=3} 2\alpha' \mathrm{Re}\left(\nabla_i \Big\langle \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot \Big(\nabla^{m'}\overline{\nabla}^{l'}(\mathrm{Rm} * \mathrm{Rm}), \nabla^{m'}\overline{\nabla}^{l'} T\Big\rangle^i\right) \\
&+ C(\epsilon')^{-1} \cdot \Big(1 + \big(\|D\mathrm{Rm}\|^2 + \|D^2T\|^2\big) \\
&\quad + \big(\|D\mathrm{Rm}\|^2 + \|D^2T\|^2\big)^2 + \big(\|D^2\mathrm{Rm}\|^2 + \|D^3T\|^2\big)\Big) \\
&+ \Big[C\epsilon' + 42a_0B\alpha\Big] \cdot \Big(\|D\mathrm{Rm}||^2 + \|D^2T\|^2\Big) \cdot \Big(\|D^2\mathrm{Rm}\|^2 + \|D^3T\|^2\Big) \\
&+ \Big[C\epsilon' + 6a_0BC_0\alpha' + 4a_0B\alpha'\Big] \cdot \Big(\|D^3\mathrm{Rm}\|^2 + \|D^4T\|^2\Big). \qquad (4.135)
\end{aligned}
$$

We choose a similar test function this time and set

$$
\begin{aligned}
G' &= \Big[\alpha' \cdot \Big(\|\mathrm{Rm}\|^2 + \|DT\|^2\Big) + \mu'\Big] \cdot \Big(\|D^2\mathrm{Rm}\|^2 + \|D^3T\|^2\Big) \\
&= (\alpha' \cdot G_0 + \mu') \cdot G_2, \qquad (4.136)
\end{aligned}
$$

where $\mu' > 0$ is a constant to be determined later. Analogous computations to before show that for $0 < \epsilon' < 1$,

$$
\begin{aligned}
\frac{\partial}{\partial t} G' \le\ & \frac{1}{2} \cdot \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot (\Delta_R G') \\
& - B^{-1}\alpha' \cdot \Big(\|D\mathrm{Rm}\|^2 + \|D^2T\|^2\Big) \cdot \Big(\|D^2\mathrm{Rm}\|^2 + \|D^3T\|^2\Big) \\
& - B^{-1}\mu' \cdot \Big(\|D^3\mathrm{Rm}\|^2 + \|D^4T\|^2\Big) \\
& + \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot 2\alpha' \\
& \quad \cdot \mathrm{Re}\Big\langle \nabla\Big(\|\mathrm{Rm}||^2 + \|DT\|^2\Big), \nabla\Big(\|D^2\mathrm{Rm}\|^2 + \|D^3T\|^2\Big)\Big\rangle \\
& + 2\mathrm{Re}\,(\mathbf{A}') + 2\mathrm{Re}\,(\mathbf{B}') + C(\epsilon')^{-1} \cdot \Big(1 + G_1 + G_1^2 + G'\Big) \\
& + \Big[C\epsilon' + 6a_0BC_0(\alpha')^2 + 84a_0BC_0^2(\alpha')^2 + 42a_0B\alpha'\mu'\Big] \\
& \quad \cdot \Big(\|D\mathrm{Rm}\|^2 + \|D^2T\|^2\Big) \cdot \Big(\|D^2\mathrm{Rm}\|^2 + \|D^3T\|^2\Big) \\
& + \Big[C\epsilon' + 12a_0BC_0^3(\alpha')^2 + 8a_0BC_0^2(\alpha')^2 + 6a_0BC_0\alpha'\mu' \\
& \quad + 4a_0B\alpha'\mu'\Big] \cdot \Big(\|D^3\mathrm{Rm}\|^2 + \|D^4T\|^2\Big), \qquad (4.137)
\end{aligned}
$$

where the terms **(A')** and **(B')** are given by

$$
\begin{aligned}
(\mathbf{A}') = {} & (\alpha')^2 \cdot \Big(\|D^2\mathrm{Rm}\|^2 + \|D^3 T\|^2\Big) \\
& \cdot \Big(\overline{\nabla}_{\bar{j}} \Big\langle \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot \big(\nabla(\mathrm{Rm} * \mathrm{Rm})\big), \mathrm{Rm} \Big\rangle^{\bar{j}} \Big) \\
& + \sum_{m'+l'=1} (\alpha')^2 \cdot \Big(\|D^2\mathrm{Rm}\|^2 + \|D^3 T\|^2\Big) \\
& \cdot \Big(\nabla_i \Big\langle \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot \big(\nabla^{m'}\overline{\nabla}^{l'}(\mathrm{Rm} * \mathrm{Rm})\big), \nabla^{m'}\overline{\nabla}^{l'} T \Big\rangle^{i} \Big), \qquad (4.138)
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{B}') = {} & \sum_{m+l=2} \alpha' \cdot \Big[\alpha' \cdot \Big(\|\mathrm{Rm}\|^2 + \|DT\|^2\Big) + \mu'\Big] \\
& \cdot \Big(\overline{\nabla}_{\bar{j}} \Big\langle \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot \big(\nabla^{m+1}\overline{\nabla}^{l}(\mathrm{Rm} * \mathrm{Rm})\big), \nabla^{m}\overline{\nabla}^{l}\mathrm{Rm} \Big\rangle^{\bar{j}} \Big) \\
& + \sum_{m'+l'=3} \alpha' \cdot \Big[\alpha' \cdot \Big(\|\mathrm{Rm}\|^2 + \|DT\|^2\Big) + \mu'\Big] \\
& \cdot \Big(\nabla_i \Big\langle \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot \big(\nabla^{m'}\overline{\nabla}^{l'}(\mathrm{Rm} * \mathrm{Rm})\big), \nabla^{m'}\overline{\nabla}^{l'} T \Big\rangle^{i} \Big). \qquad (4.139)
\end{aligned}
$$

Proceeding as before, we get that for $p \geq 3$,

$$
\begin{aligned}
\frac{\partial}{\partial t}\Big(\int_X (G')^p\Big) \leq {} & C(\epsilon')^{-1} \cdot \int_X \Big(1 + G_1^p + G_1^{2p} + (G')^p\Big) \\
& + \Big[C\epsilon' + 6a_0BC_0^2(\alpha')^2(\mu')^{-1}p(p-1) + 56a_0B\alpha' p(p-1) \\
& \quad - B^{-1}p(p-1)\Big] \cdot \int_X (G')^{p-2} \cdot \|\overline{\nabla}G'\|^2 \\
& + \Big[C\epsilon' + 252a_0BC_0^2(\alpha')^2 p + 90a_0BC_0^2(\alpha')^2 p(p-1) \\
& \quad + 510a_0BC_0(\alpha')^2 p + 42a_0B\alpha'\mu' p^2 - \frac{1}{2}B^{-1}\alpha' p\Big] \\
& \quad \cdot \int_X (G')^{p-1} \cdot \Big(\|D\mathrm{Rm}\|^2 + \|D^2T\|^2\Big) \cdot \Big(\|D^2\mathrm{Rm}\|^2 + \|D^3T\|^2\Big) \\
& + \Big[C\epsilon' + 28a_0BC_0^4(\alpha')^2 p(p-1) + 16000B^3C_0^2\alpha' p + 12a_0BC_0^3(\alpha')^2 p \\
& \quad + 176a_0BC_0^2(\alpha')^2 p + 14a_0BC_0^2\alpha'\mu' p(p-1) + 6a_0BC_0\alpha'\mu' p \\
& \quad + 4a_0B\alpha'\mu' p - B^{-1}\mu' p\Big] \cdot \int_X (G')^{p-1} \cdot \Big(\|D^2\mathrm{Rm}\|^2 + \|D^3T\|^2\Big).
\end{aligned}
\qquad (4.140)
$$

We can read off the required inequalities from the terms to find a bound on $\alpha'$ and choose $\mu'$ appropriately. As in §4.3.1, we also require $\int_X G_1^{2p}$ to be uniformly bounded and must have $\alpha'$ satisfy an improved bound from Theorem 4.3.3. Similar reasoning then yields

**Corollary 4.3.4.** *Let $k = 1$ and set $G_q = \|D^q \mathrm{Rm}\|^2 + \|D^{q+1}T\|^2$. Suppose the assumptions (4.14) - (4.16) hold and set*

$$\begin{aligned} G' &= \left[\alpha' \cdot \left(\|\mathrm{Rm}\|^2 + \|D^2 T\|^2\right) + \mu'\right] \cdot \left(\|D^2\mathrm{Rm}\|^2 + \|D^3 T\|^2\right) \\ &= (\alpha' \cdot G_0 + \mu) \cdot G_2 \end{aligned} \tag{4.141}$$

*for some $\mu' > 0$.*

*i) If $p \geq 3$ and*

$$\alpha' < \frac{1}{10^7 a_0 B^6 \max(1, C_0)^2 p}, \tag{4.142}$$

*then for*

$$\mu' = \frac{1}{100 a_0 B^2 p} \tag{4.143}$$

*there exists some constant $\Lambda'_p = \Lambda'_p(k, \alpha') > 0$ such that*

$$\int_X (G')^p(t) \leq \left(1 + \int_X (G')^p(0)\right) e^{\Lambda'_p t} < \left(1 + \int_X (G')^p(0)\right) e^{\Lambda'_p \tau}. \tag{4.144}$$

*That is, $\int_X (G')^p(t)$ is uniformly bounded along the flow.*

*In particular, we get that $\int_X G_2^p(t)$ and both*

$$\left(\int_X \|D^2\mathrm{Rm}(t)\|^{2p}\right)^{\frac{1}{2p}} \text{ and } \left(\int_X \|D^3 T(t)\|^{2p}\right)^{\frac{1}{2p}} \tag{4.145}$$

*are bounded along the Anomaly flow;*

*ii) If instead $p \in [1, 3)$ and*

$$\alpha' < \frac{1}{3 \cdot 10^7 a_0 B^6 \max(1, C_0)^2}, \tag{4.146}$$

*then for*

$$\mu' = \frac{1}{300 a_0 B^2}, \tag{4.147}$$

*the function* $\int_X (G')^p(t)$ *and also* $\int_X G_2^p(t)$ *and both*

$$\left(\int_X \|D^2\mathrm{Rm}(t)\|^{2p}\right)^{\frac{1}{2p}} \text{ and } \left(\int_X \|D^3T(t)\|^{2p}\right)^{\frac{1}{2p}} \tag{4.148}$$

*are uniformly bounded along the Anomaly flow.*

Using the Sobolev Embedding Theorem and inductive bootstrapping argument from §4.2.4, we can conclude the following:

**Theorem 4.3.5.** *Suppose that there exist positive constants* $B, C_0$ *such that*

$$B^{-1} \leq \left(\frac{1}{2\|\Upsilon\|_\omega}\right) \leq B, \tag{4.149}$$

$$\|T\|, \|\overline{T}\|, \|\mathrm{Rm}\|, \|DT\|, \|D\overline{T}\| \leq C_0 \tag{4.150}$$

*along the Anomaly flow on* $[0, \tau)$. *If*

$$\alpha' < \frac{1}{3 \cdot 10^7 a_0 B^6 \max(1, C_0)^2}, \tag{4.151}$$

*then there exist positive constants* $C_q$ *for* $q \geq 1$ *such that*

$$\|D^q\mathrm{Rm}\|, \|D_{q+1}T\|, \|D^{q+1}\overline{T}\| \leq C_q \tag{4.152}$$

*along the Anomaly flow. These constants* $C_q$ *for* $q \geq 1$ *depend on* $\tau$, *the initial metric* $g_0$, *the slope parameter* $\alpha'$, *and the initial bounds* $B$ *and* $C_0$.

**Remark 4.3.6.** Instead of our choice of coupled upper and lower bounds

$$B^{-1} \leq \left(\frac{1}{2\|\Upsilon\|_\omega}\right) \leq B, \tag{4.153}$$

we could have instead chosen independent bounds

$$B_{\min} \leq \left(\frac{1}{2\|\Upsilon\|_\omega}\right) \leq B_{\max}. \tag{4.154}$$

In this case, our derived bounds on $\alpha'$ from Theorem 4.3.5 (the $k = 1$ case) and Corollary 4.3.2 (the $k \geq 2$ case) respectively become

$$\alpha' < \frac{\min(1, B_{\min})^3}{3 \cdot 10^7 a_0 \max(1, B_{\max})^3 \max(1, C_0)^2}, \quad (k = 1), \tag{4.155}$$

$$\alpha' < \frac{B_{\min}}{26 a_0 B_{\max} C_0}, \quad (k \geq 2). \tag{4.156}$$

In particular, these derived bounds can respectively be rewritten as

$$\alpha' \cdot \|\mathrm{Rm}\|^2 < \Pi_1, \quad (k = 1), \tag{4.157}$$

$$\alpha' \cdot \|\mathrm{Rm}\| < \Pi_2, \quad (k \geq 2) \tag{4.158}$$

for some dimensionless constants $\Pi_1$ and $\Pi_2$. This is intriguing as the units on the LHS of the previous two equations differ but the RHS of both are dimensionless.

## 4.4 Long-time Existence

Now that we have established the relevant $L^\infty$-bounds for covariant derivatives of curvature and torsion, we can appeal to the argument in [PPZ18b] to obtain long-time existence of the Anomaly flow. We outline the argument below.

### 4.4.1 $C^0$- and $C^1$-Bounds

Suppose that there exist positive constants $B$, $C_0$ such that

$$B^{-1} \leq \left( \frac{1}{2\|\Upsilon\|_\omega} \right) \leq B, \tag{4.159}$$

$$\|T\|, \|\overline{T}\|, \|\mathrm{Rm}\|, \|DT\|, \|D\overline{T}\| \leq C_0 \tag{4.160}$$

along the Anomaly flow on $[0, \tau)$. The metrics $g$ are uniformly equivalent close to time $\tau$, and so if $\widehat{g}$ is a fixed reference metric, the endomorphism

$$h^\alpha{}_\beta = (\widehat{g})^{\alpha\overline{\gamma}} g_{\overline{\gamma}\beta} \tag{4.161}$$

has a uniform $C^0$-bound.

We have that the Chern curvatures of $\widehat{g}$ and $g$ are related by

$$\widehat{R}_{\overline{p}q}{}^\alpha{}_\beta - R_{\overline{p}q}{}^\alpha{}_\beta = \partial_{\overline{p}}(h^\alpha{}_\gamma \widehat{\nabla}_q h^\gamma{}_\beta). \tag{4.162}$$

Since Rm is bounded, we have a $C^1$-bound on the endomorphism $h$.

### 4.4.2 $C^k$-Bounds

For each $q \geq 0$ and for a tensor $A$, set

$$\|\widehat{D}^q A\|^2 = \sum_{m+l=q} \|\widehat{\nabla}^m \overline{\widehat{\nabla}}^l A\|^2. \tag{4.163}$$

We also set the tensor $S$ to be the difference of the Christoffel symbols of the reference and evolving metrics:

$$S^\alpha{}_{k\beta} = \widehat{\Gamma}^\alpha{}_{k\beta} - \Gamma^\alpha{}_{k\beta} = -g^{\alpha\overline{\gamma}}\widehat{\nabla}_k g_{\overline{\gamma}\beta}. \tag{4.164}$$

From Theorem 4.3.5, we see that if $\alpha'$ is sufficiently small, the all covariant derivatives of Rm and $T$ with respect to the moving metric are uniformly bounded on $[0,\tau)$. Since

$$\frac{\partial}{\partial t} g = \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \cdot \Big[\mathrm{Rm} + T * \overline{T} + \alpha' \cdot \big(\mathrm{Rm} * \mathrm{Rm} + \Phi\big)\Big], \tag{4.165}$$

it follows that all covariant derivatives of $\frac{\partial}{\partial t} g$ with respect to the evolving metric are also uniformly bounded.

We also have the following from [PPZ18b]:

**Proposition 4.4.1** (Phong–Picard–Zhang [PPZ18b] Proposition 2)**.** *Suppose all covariant derivatives of* Rm *and* $T$ *with respect to the evolving metric are uniformly bounded along the Anomaly flow on* $[0,\tau)$*. If for* $k \geq 1$ *there exist positive constants* $C'_0, C'_1, \ldots, C'_q$ *such that*

$$\|D^q S\|, \|\widehat{D}^{q+1} g\| \leq C'_q \text{ for } 1 \leq q \leq k-1, \tag{4.166}$$

$$\|g\|, \|S\|, \|\widehat{D} g\| \leq C'_0 \tag{4.167}$$

*along the Anomaly flow, then there exists some positive* $C'_k$ *such that*

$$\|D^k S\|, \|\widehat{D}^{k+1} g\| \leq C'_k. \tag{4.168}$$

The $C^0$- and $C^1$-bounds from §4.4.1 imply the existence of some positive $C'_0$ such that

$$\|g\|, \|S\|, \|\widehat{D} g\| \leq C'_0. \tag{4.169}$$

Inductively applying Proposition 4.4.1, we see that the covariant derivatives of $g$ with respect to the reference metric $\widehat{g}$, that is $\|\widehat{D}^q g\|$ for each $q \geq 0$, are all uniformly bounded along the flow.

Now, for any $i, m, l$, we have

$$\begin{aligned}\frac{\partial^i}{\partial t^i}(\widehat{\nabla}^m \overline{\widehat{\nabla}}^l g) &= \widehat{\nabla}^m \overline{\widehat{\nabla}}^l \Big(\frac{\partial^i}{\partial t^i} g\Big) \\ &= \widehat{\nabla}^m \overline{\widehat{\nabla}}^l \frac{\partial^{i-1}}{\partial t^{i-1}} \Bigg( \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \\ &\qquad \cdot \Big[ \mathrm{Rm} + T * \overline{T} + \alpha' \cdot \big(\mathrm{Rm} * \mathrm{Rm} + \Phi\big)\Big]\Bigg). \end{aligned} \tag{4.170}$$

The time derivatives of $\frac{\partial}{\partial t} g$ can also be expressed in terms of time derivatives of connections, Rm, and $T$, which our calculations in previous sections have shown to be bounded. As such

$$\frac{\partial^i}{\partial t^i}(\widehat{\nabla}^m \overline{\widehat{\nabla}}^l g) \tag{4.171}$$

is uniformly bounded along the flow on $[0, \tau)$. We can thus extend the Anomaly flow smoothly to time $t = \tau$, and by the short-time existence of the Anomaly flow from Theorem 2 of [PPZ18c], we can further extend it to $[0, \tau + \epsilon)$ for some $\epsilon > 0$.

We thus have the following:

**Theorem 4.4.2.** *Suppose that there exist positive constants $B$, $C_0$ such that*

$$B^{-1} \leq \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \leq B. \tag{4.172}$$

$$\|T\|, \|\overline{T}\|, \|\mathrm{Rm}\|, \|DT\|, \|D\overline{T}\| \leq C_0 \tag{4.173}$$

*along the Anomaly flow on $[0, \tau)$. If*

$$\alpha' < \frac{1}{3 \cdot 10^7 a_0 B^6 \max(1, C_0)^2}, \tag{4.174}$$

*then the flow can be extended to $[0, \tau + \epsilon)$ for some $\epsilon > 0$.*

# Part II

# $G_2$-Geometry

# Chapter 5

# Introduction

The rest of this thesis deals with manifolds with $G_2$-structure. We briefly cover some basics and extend some of the ideas from §1 to this setting.

## 5.1 Manifolds with $G_2$-Structure

**Definition 5.1.1.** A 3-form on a 7-dimensional manifold $M$ is called non-degenerate if for any point $p \in M$ and any $0 \neq Y \in T_pM$,

$$(Y \lrcorner \varphi) \wedge (Y \lrcorner \varphi) \wedge \varphi \neq 0. \tag{5.1}$$

A smooth non-degenerate 3-form is also called a $G_2$-structure. In this case, there is a unique metric $g$ and orientation such that if vol is the volume form associated to the metric and orientation, then for any $p \in M$ and $Y, Z \in T_pM$, we have

$$-\frac{1}{6}(Y \lrcorner \varphi) \wedge (Z \lrcorner \varphi) \wedge \varphi = g(Y, Z)\mathrm{vol}. \tag{5.2}$$

With these induced structures, we also get a Hodge star operator $\star$ and a dual 4-form $\psi = \star\varphi$.

**Remark 5.1.2.** Some authors choose to use the other orientation on $M$ by flipping the sign in (5.2).

**Remark 5.1.3.** At times, we may (as an abuse of notation) also refer to the dual 4-form $\psi$ as a $G_2$-structure.

One can think of a $G_2$-structure as a smooth pointwise identification of the tangent space $T_pM$ with the imaginary octonions $\mathrm{Im}\,\mathbb{O}$.

On $\operatorname{Im}\mathbb{O} \simeq \mathbb{R}^7$, we have the commutator $[\cdot,\cdot] : \operatorname{Im}\mathbb{O} \times \operatorname{Im}\mathbb{O} \to \operatorname{Im}\mathbb{O}$ given by

$$[a, b] = ab - ba, \tag{5.3}$$

and the associator $[\cdot,\cdot,\cdot] : \operatorname{Im}\mathbb{O} \times \operatorname{Im}\mathbb{O} \times \operatorname{Im}\mathbb{O} \to \operatorname{Im}\mathbb{O}$ given by

$$[a, b, c] = (ab)c - a(bc). \tag{5.4}$$

There is also a cross-product $\times : \operatorname{Im}\mathbb{O} \times \operatorname{Im}\mathbb{O} \to \operatorname{Im}\mathbb{O}$ defined by

$$a \times b = \operatorname{Im}(ab). \tag{5.5}$$

The group $G_2$ is the subgroup of $\mathrm{GL}(7, \mathbb{R})$ that preserves these structures along with the standard Euclidean metric and volume form.

In this way, we get the 3-form $\varphi$ on $M$ from the commutator $[\cdot,\cdot]$ and usual inner product $\langle\cdot,\cdot\rangle$:

$$\varphi(a, b, c) = \frac{1}{2}\langle a, [b, c]\rangle = \frac{1}{2}\langle [a, b], c\rangle. \tag{5.6}$$

Similarly, the 4-form $\psi$ is derived from the associator $[\cdot,\cdot,\cdot]$:

$$\psi(a, b, c, d) = \frac{1}{2}\langle a, [b, c, d]\rangle = -\frac{1}{2}\langle [a, b, c], d\rangle. \tag{5.7}$$

Finally, the cross-product $\times$ is realized by the relation

$$\varphi(a, b, c) = \langle a \times b, c\rangle. \tag{5.8}$$

We note here that a 7-dimensional manifold $M$ admits a $G_2$-structure if and only if it is orientable and spinnable.

### Decomposition of Forms

Given a manifold $M$ with $G_2$-structure $\varphi$, the bundles $\Omega^k(M)$ of $k$-forms for $0 \le k \le 7$ decompose fiberwise into irreducible representations of the group $G_2$. The space $\Omega^k(M)$ is itself irreducible when $k = 0, 1, 6, 7$. For $k = 2, 3$, we have

$$\Omega^2(M) = \Omega^2_7(M) \oplus \Omega^2_{14}(M), \tag{5.9}$$

$$\Omega^3(M) = \Omega^3_1(M) \oplus \Omega^3_7(M) \oplus \Omega^3_{27}(M), \tag{5.10}$$

where the spaces $\Omega^k_l$ has pointwise dimension $l$. These irreducible representations can be described invariantly as

$$\Omega^2_7(M) = \{Y \lrcorner \varphi \mid Y \in \mathfrak{X}(M)\} = \{\beta \in \Omega^2(M) \mid \star(\beta \wedge \varphi) = -2\beta\}, \tag{5.11}$$

$$\begin{aligned}\Omega^2_{14}(M) &= \{\beta \in \Omega^2(M) \mid \beta \wedge \psi = 0\} \\ &= \{\beta \in \Omega^2(M) \mid \star(\beta \wedge \varphi) = \beta\} \simeq \mathfrak{g}_2,\end{aligned} \tag{5.12}$$

and

$$\Omega^3_1(M) = \{f\varphi \mid f \in C^\infty(M)\}, \tag{5.13}$$

$$\Omega^3_7(M) = \{Y \lrcorner \psi \mid Y \in \mathfrak{X}(M)\}, \tag{5.14}$$

$$\Omega^3_{27}(M) = \{\gamma \in \Omega^3(M) \mid \gamma \wedge \varphi = \gamma \wedge \psi = 0\}. \tag{5.15}$$

The decomposition for the spaces of 4- and 5-forms are obtained from these using the Hodge star operator $\star$.

We will not require much else about the decomposition of forms on a manifold with $G_2$-structure in this thesis. As such, we refer the interested reader to [Kar09, Kar20] for more details.

### Torsion of a $G_2$ Structure

The decomposition of forms described above can be applied to the 4-form $d\varphi$ and the 5-form $d\psi$. This will in turn allow us to define the torsion forms of the $G_2$-structure $\varphi$.

**Definition 5.1.4.** The torsion forms of a $G_2$-structure $\varphi$ are

$$\tau_0 \in \Omega^0, \quad \tau_1 \in \Omega^1, \quad \tau_2 \in \Omega^2_{14}, \quad \tau_3 = \Omega^3_{27}, \tag{5.16}$$

and are defined by the equations

$$d\varphi = \tau_0 \psi + 3\tau_1 \wedge \varphi + \star\tau_3, \tag{5.17}$$

$$d\psi = 4\tau_1 \wedge \psi + \star\tau_2. \tag{5.18}$$

**Remark 5.1.5.** While it is not immediately clear from the above equations, it can be shown that the $\tau_1$ appearing in both (5.17) and (5.18) are the same. For a proof of this, see [Kar09].

Using identities from the decomposition, we have expressions for the forms $\tau_0$ and $\tau_1$ given by

$$\tau_0 = \frac{1}{7} \star (\varphi \wedge d\varphi), \tag{5.19}$$

$$\tau_1 = \frac{1}{12} \star \big(\varphi \wedge \star(d\varphi)\big) = \frac{1}{12} \star \big(\psi \wedge \star(d\psi)\big). \tag{5.20}$$

Using the torsion forms, we can define 16 classes of $G_2$-structure based on which torsion forms are non-zero. We list a few special cases of particular interest to us.

**Definition 5.1.6.** A $G_2$-structure $\varphi$ is called

i) integrable if $\tau_2 = 0$,

ii) closed if $d\varphi = 0$ (and hence $\tau_0 = \tau_1 = \tau_3 = 0$),

iii) coclosed if $d\psi = 0$ (and hence $\tau_1 = \tau_2 = 0$),

iv) nearly parallel if $\tau_1 = \tau_2 = \tau_3 = 0$ (or equivalently $d\varphi = \lambda\psi$ for some locally constant $\lambda$),

v) torsion-free if it is both closed and coclosed.

We note that if $\varphi$ is torsion-free, then the metric $g$ that it induces is Ricci-flat and has holonomy group $\mathrm{Hol}(g)$ contained in the group $G_2$.

## 5.2 The $G_2$-Strominger System and $G_2$-Anomaly Flow

We recall the Hull–Strominger system (1.22) - (1.25) for a Calabi–Yau threefold. Common geometric features in 6 and 7 dimensions have allowed for a generalization of this system to the $G_2$ setting [CGFT22, dlOLS18, FIUV15].

**Definition 5.2.1.** Let $M$ be a compact 7-manifold admitting $G_2$ structures and let $E \to M$ be a vector bundle. Fix a constant $\alpha' > 0$. A solution to the $G_2$-Strominger system with slope parameter $\alpha'$ consists of a $G_2$-structure $\varphi$ on $M$ and connections $A$ on $E$ and $\nabla$ on $M$ such that

$$F_A \wedge \psi = 0, \tag{5.21}$$

$$dH - \alpha' \cdot \Big(\mathrm{tr}(F_A \wedge F_A) - \mathrm{tr}(R_\nabla \wedge R_\nabla)\Big) = 0, \tag{5.22}$$

$$d\big(e^{-2f}\psi\big) = 0, \tag{5.23}$$

for some function $f$ called the dilaton, where the 3-form $H$ is given by

$$H = \frac{1}{6}\tau_0\varphi + \star(\tau_1 \wedge \varphi) - \tau_3, \tag{5.24}$$

and $R_\nabla$ and $F_A$ are curvatures of $\nabla$ and $A$ respectively.

Here, the first condition is that both $\nabla$ and $A$ are instanton connections, the second condition is the analogue of the heterotic Bianchi identity, while the third is a conformally coclosed condition with $e^{-2f}$ playing the role of $\|\Upsilon\|_\omega$ in the Calabi–Yau case. In particular, one can check that this final condition implies that

$$d\psi = 2df \wedge \psi \tag{5.25}$$

and so

$$\tau_1 = \frac{1}{2}df \text{ and } \tau_2 = 0. \tag{5.26}$$

As such, a solution to the $G_2$-Strominger system must involve an integrable $G_2$-structure.

**Remark 5.2.2.** Like the Hull–Strominger system, there are varying conventions with the connections involved. For example, some authors impose the condition that the connection $\nabla$ be the Levi–Civita connection $\nabla_\varphi$ associated to the metric $g$ induced by $\varphi$.

As it was in the Calabi–Yau setting, this should again generalize the case where we have a torsion-free structure. Indeed, suppose we have a torsion-free $G_2$-structure $\varphi$ on $M$. Then, by setting $E$ to be the tangent bundle $TM$ and both $A$ and $\nabla$ to be the Levi–Civita connection $\nabla_\varphi$ associated to $\varphi$ (via its induced metric $g$), we see that both $f$ and $H$ must vanish and hence both (5.22) and (5.23) are satisfied. For holonomy reasons, we must have that $F_A = R_\nabla \in \mathcal{S}^2\mathfrak{g}_2 \simeq \mathcal{S}^2\Omega^2_{14}$. As such, by (5.12), both $A$ and $\nabla$ are instanton connections and the full system is satisfied.

The Anomaly flow also has a $G_2$-analogue (for the $\alpha' = 0$ case) which was proposed by Ashmore–Minasian–Proto [AMP24]. This is a geometric flow on a conformally coclosed $G_2$-structure $\varphi$ with evolution equation given by

$$\frac{\partial}{\partial t}(e^{-2f}\psi) = -dH. \tag{5.27}$$

As was the case in the Calabi–Yau setting, this flow can be seen to preserve the conformally coclosed condition. In addition, to solve the full $G_2$-Strominger system, this flow should be coupled with flows on the connections $A$ and $\nabla$.

At the time of writing, there is still much that is unknown about the $G_2$-Strominger system and $G_2$-Anomaly flow. For the remainder of this thesis, we instead mainly focus on torsion-free $G_2$-structures and other related geometric flows.

## 5.3 $G_2$-Structures from SU(3)-Structures

The inclusion $\mathrm{SU}(3) \subseteq G_2$ suggests a strong relation between SU(3)-structures and $G_2$-structures. In this section, we describe a manner in which we can obtain $G_2$-structures from SU(3)-structures by considering $S^1$-fibrations.

**Remark 5.3.1.** We note the inclusion $\mathrm{SU}(2) \subseteq G_2$ and that similar constructions can be done with SU(2)-structures and $T^3$-fibrations (see *e.g.,* [FY18, PS24]), however, we will not consider those here.

### 5.3.1 $G_2$-Structures on Trivial $S^1$-Fibrations

Let $X$ be a Kähler Calabi–Yau threefold and $(\omega, \Upsilon)$ be a SU(3)-structure on $X$ with Kähler form $\omega$. Both $\omega$ and $\Upsilon$ are closed forms with local descriptions given by

$$\omega = \sqrt{-1} g_{j\overline{k}} dz^j \wedge d\overline{z}^{\overline{k}}, \tag{5.28}$$

$$\Upsilon = f dz^1 \wedge dz^2 \wedge dz^3, \tag{5.29}$$

were $g = g_{j\overline{k}}$ is the metric associated to $\omega$ and $f$ is a local holomorphic function.

The pair $(\omega, \Upsilon)$ satisfies the relations

$$\frac{\omega^3}{3!} = \mathrm{vol} = \frac{\sqrt{-1}}{\|\Upsilon\|_\omega^2} \Upsilon \wedge \overline{\Upsilon} = 2\mathrm{Re}\left(\frac{1}{\|\Upsilon\|_\omega}\Upsilon\right) \wedge \mathrm{Im}\left(\frac{1}{\|\Upsilon\|_\omega}\Upsilon\right). \tag{5.30}$$

In addition, we also have

$$\star^2\beta = (-1)^k\beta \text{ for } \beta \in \Omega^k(X), \quad \star\mathrm{Re}\,(\Upsilon) = \mathrm{Im}\,(\Upsilon), \quad \star\omega = \frac{1}{2}\omega^2. \tag{5.31}$$

Let $r$ denote the angle coordinate on $S^1$ so that $dr$ is the globally defined volume form on $S^1$ with respect to the standard round metric. If $F$ is a smooth nowhere-vanishing complex function on $X$ and $G$ is a smooth strictly positive function on $X$, we can consider the 3-form $\varphi$ on $M = S^1 \times X$ given by

$$\varphi = \mathrm{Re}\left(\frac{F}{\|\Upsilon\|_\omega}\Upsilon\right) - G dr \wedge \omega. \tag{5.32}$$

As seen in [KMT12], this form is non-degenerate and thus is a $G_2$-structure on $M$. The 3-form $\varphi$ induces the metric

$$g_\varphi = 4|F|^{-\frac{4}{3}} G^2 dr \otimes dr + \frac{1}{2}|F|^{\frac{2}{3}} g, \tag{5.33}$$

with associated volume form

$$\mathrm{vol}_{\varphi} = \frac{1}{4}|F|^{\frac{4}{3}} G dr \wedge \mathrm{vol}. \tag{5.34}$$

More details about this can be found in Appendix D.

Using the expressions (5.33) and (5.34), one can compute that for $\beta \in \Omega^k(X)$, the Hodge star $\star_{\varphi}$ on $M$ acts by

$$\star_{\varphi}\beta = (-1)^k 2^{(-2+k)} |F|^{(\frac{4}{3}-\frac{2}{3}k)} G dr \wedge (\star\beta), \tag{5.35}$$

$$\star_{\varphi}(dr \wedge \beta) = 2^{(-4+k)} |F|^{(\frac{8}{3}-\frac{2}{3}k)} G^{-1} (\star\beta). \tag{5.36}$$

As such, we can check that the dual 4-form $\psi$ is given by

$$\psi = -2|F|^{-\frac{2}{3}} G dr \wedge \mathrm{Im}\left(\frac{F}{\|\Upsilon\|_{\omega}}\Upsilon\right) - \frac{1}{8}|F|^{\frac{4}{3}}\omega^2. \tag{5.37}$$

**Remark 5.3.2.** We note that the factors of 2 appearing in the above expressions are an artifact of the relation between a Hermitian metric $g_{j\bar{k}}$ and its associated Riemannian metric $g_{jk}$ (see Appendix D).

Certain choices for the functions $F$ and $G$ result in particular types of $G_2$-structures. In particular, if we set $F = \|\Upsilon\|_{\omega}$ and $G = 1$, then we get that

$$\varphi = \mathrm{Re}\,(\Upsilon) - dr \wedge \omega, \tag{5.38}$$

which is a closed 3-form. If instead, we reverse our choices for $F$ and $G$, (that is, set $F = 1$ and $G = \|\Upsilon\|_{\omega}$), we see that

$$\psi = -2dr \wedge \mathrm{Im}\,(\Upsilon) - \frac{1}{8}\omega^2, \tag{5.39}$$

and so the $G_2$-structure we obtain is coclosed.

### 5.3.2 $G_2$-Structures on Contact Calabi–Yau 7-Folds

We now consider the case where the $S^1$-fibration is non-trivial. In particular, we study structures on contact Calabi–Yau 7-folds, which admit a smoothly varying family of codimension 1 subspaces of the tangent space at each point which satisfy a non-integrability condition.

**Definition 5.3.3.** A contact Calabi–Yau (cCY) 7-fold consists of a quadruple $(M, \eta, \Phi, \Upsilon)$ such that

i) $M$ is a 7-dimensional Sasakian manifold with Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ and vanishing first basic Chern class $c_1^B(M) = 0$ (see Appendix C);

ii) $\Upsilon$ is a closed nowhere-vanishing transverse form on $\mathcal{D} = \ker \eta$ of type $(3, 0)$ with

$$\frac{\omega^3}{3!} = \mathrm{vol}_{\mathcal{D}} = \frac{\sqrt{-1}}{\|\Upsilon\|_\omega^2} \Upsilon \wedge \overline{\Upsilon} = 2\mathrm{Re}\left(\frac{1}{\|\Upsilon\|_\omega}\Upsilon\right) \wedge \mathrm{Im}\left(\frac{1}{\|\Upsilon\|_\omega}\Upsilon\right), \quad (5.40)$$

where $\omega = d\eta$.

We refer to the pair $(\omega, \Upsilon)$ as a transverse SU(3)-structure. As with the regular case, the norm $\|\Upsilon\|_\omega$ is constant when $\omega$ is transverse Ricci-flat.

**Remark 5.3.4.** A contact Calabi–Yau manifold $(M, \eta, \Phi, \Upsilon)$ has a transverse Calabi–Yau geometry on the distribution $\mathcal{D} = \ker \eta$, in the sense of foliations, given by $g|_{\mathcal{D}}, \omega$, and $\Upsilon$. When the Sasakian structure is regular, or quasi-regular, $M$ is an $S^1$-(orbi)bundle over a Calabi–Yau orbifold $Z = M/\mathcal{F}_\xi$, where $\mathcal{F}_\xi$ is the foliation obtained from the Reeb vector field $\xi$. The Sasakian geometry can also be irregular, and in this case, there is no $S^1$-fibration structure on $M$ compatible with the contact Calabi–Yau geometry.

In this case, we again have the local descriptions of $\omega = d\eta$ and $\Upsilon$

$$\omega = \sqrt{-1} g_{j\bar{k}} dz^j \wedge \overline{z}^{\bar{k}}, \quad (5.41)$$

$$\Upsilon = f dz^1 \wedge dz^2 \wedge dz^3 \quad (5.42)$$

where $g_{j\bar{k}}$ and $f$ are basic functions.

Using the basic Hodge star operator $\star_B$, we have the relations

$$(\star_B)^2 \beta = (-1)^k \beta \text{ for } \beta \in \Omega_B^k(M), \quad \star_B \mathrm{Re}\,(\Upsilon) = \mathrm{Im}\,(\Upsilon), \quad \star_B \omega = \frac{1}{2}\omega^2. \quad (5.43)$$

We use a similar construction to 5.3.1 in order obtain coclosed $G_2$-structure a cCY 7-fold. Define a 3-form by

$$\varphi = \mathrm{Re}\left(\frac{1}{\|\Upsilon\|_\omega}\Upsilon\right) - \|\Upsilon\|_\omega \cdot \eta \wedge \omega. \quad (5.44)$$

This induces the metric

$$g_\varphi = 4\|\Upsilon\|_\omega^2 \cdot \eta \otimes \eta + \frac{1}{2} g|_{\mathcal{D}} \quad (5.45)$$

and volume form

$$\mathrm{vol}_\varphi = \frac{1}{4}\|\Upsilon\|_\omega \cdot \eta \wedge \mathrm{vol}|_{\mathcal{D}}, \text{ where } \mathrm{vol}|_{\mathcal{D}} = \frac{\omega^3}{3!}. \tag{5.46}$$

The induced Hodge star $\star_\varphi$ also has a similar relation to the basic Hodge star

$$\star_\varphi \beta = (-1)^k 2^{(-2+k)}\|\Upsilon\|_\omega \cdot \eta \wedge (\star_B \beta), \tag{5.47}$$

$$\star_\varphi (\eta \wedge \beta) = 2^{(-4+k)}\frac{1}{\|\Upsilon\|_\omega} \cdot (\star_B \beta) \tag{5.48}$$

where $\beta \in \Omega^k_B(M)$.

Hence our dual 4-form is

$$\psi = -2\eta \wedge \mathrm{Im}\,(\Upsilon) - \frac{1}{8}\omega^2. \tag{5.49}$$

**Remark 5.3.5.** We note that there is no similar construction for closed $G_2$-structures. This is because the contact form $\eta$ is not closed and also since

$$d(\eta \wedge \omega) = d\eta \wedge \omega = \omega^2 \neq 0. \tag{5.50}$$

**Remark 5.3.6.** The sign convention used for the $G_2$-structure here is opposite to that used in [SESS24] to match that of the previous section and also [PS24]. Additionally, we do not scale the metric here which will result in the factors of 2 appearing (as described in Remark 5.3.2).

# Chapter 6

# The Laplacian Flow

Having constructed $G_2$-structures from SU(3)-structures, we are interested to see how the underlying structures change under deformations of the constructed ones. In this chapter, we study Bryant's Laplacian flow [Bry06] on a trivial $S^1$-fibration and see how it induces a particular flow on the base structures.

## 6.1 Properties of the Laplacian Flow

The Laplacian flow is a geometric flow on a manifold with $G_2$-structure. It evolves a $G_2$-structure $\varphi$ by

$$\frac{\partial}{\partial t}\varphi = \Delta_{d,\varphi}\varphi. \tag{6.1}$$

Here $\Delta_{d,\varphi} = dd^*_\varphi + d^*_\varphi d$ is the Hodge Laplacian. We note that since the metric is dependent on the moving $G_2$-structure $\varphi$, the Hodge Laplacian also depends on $\varphi$ as implied by the notation.

If the $G_2$-structure is closed, this reduces to

$$\frac{\partial}{\partial t}\varphi = dd^*_\varphi\varphi, \tag{6.2}$$

and so the flow preserves closedness given an initial closed $G_2$-structure $\varphi_0$. In this case, the flow has also been shown to have short-time existence and uniqueness [BX11] (see also [BV20, Gri13, Lot20]).

The stationary points of the Laplacian flow are torsion-free $G_2$-structures. In the case where $M$ is compact, this follows from an integration-by-parts argument. (For a proof in the non-compact case, see [DGK21].)

## 6.2 Evolution Equations for the Base Structures

We apply the Laplacian flow to our closed $G_2$ Ansatz on $M = S^1 \times X$ from §5.3.1. Recall that we defined $\varphi$ by

$$\varphi = \mathrm{Re}\,(\Upsilon) - dr \wedge \omega \tag{6.3}$$

where $\omega$ is Kähler form and $\Upsilon$ is a nowhere-vanishing holomorphic $(3,0)$-form on a Calabi–Yau threefold $X$.

From this, we have the induced metric, volume form, and dual 4-form given respectively by

$$g_\varphi = 4\|\Upsilon\|_\omega^{-\frac{4}{3}} \cdot dr \otimes dr + \frac{1}{2}\|\Upsilon\|_\omega^{\frac{2}{3}} \cdot g, \tag{6.4}$$

$$\mathrm{vol}_\varphi = \frac{1}{4}\|\Upsilon\|_\omega^{\frac{4}{3}} \cdot dr \wedge \mathrm{vol}, \tag{6.5}$$

$$\psi = -2\|\Upsilon\|_\omega^{-\frac{2}{3}} \cdot dr \wedge \mathrm{Im}\,(\Upsilon) - \frac{1}{8}\|\Upsilon\|_\omega^{\frac{4}{3}} \cdot \omega^2. \tag{6.6}$$

The Hodge star also then acts by

$$\star_\varphi \beta = (-1)^k 2^{(-2+k)} |F|^{(\frac{4}{3} - \frac{2}{3}k)} G dr \wedge (\star\beta), \tag{6.7}$$

$$\star_\varphi (dr \wedge \beta) = 2^{(-4+k)} |F|^{(\frac{8}{3} - \frac{2}{3}k)} G^{-1} (\star\beta) \tag{6.8}$$

for $\beta \in \Omega^k(X)$.

To apply the Laplacian flow, we need to compute the Hodge Laplacian of $\varphi$.

**Lemma 6.2.1.** *If $\varphi$ is the $G_2$ structure defined by* (6.3)*, then*

$$\Delta_{d,\varphi}\varphi = 2\mathcal{L}_{\nabla(\|\Upsilon\|_\omega^{-\frac{2}{3}})}\Big( - \mathrm{Re}\,(\Upsilon) - 2dr \wedge \omega\Big). \tag{6.9}$$

*Proof.* Since $\varphi$ is closed 3-form on a 7-manifold, we have

$$\Delta_{d,\varphi}\varphi = dd_\varphi^* \varphi = -d \star_\varphi d\psi. \tag{6.10}$$

We then compute that

$$\begin{aligned} d\psi &= d\Big( - 2\|\Upsilon\|_\omega^{-\frac{2}{3}} \cdot dr \wedge \mathrm{Im}\,(\Upsilon) - \frac{1}{8}\|\Omega\|^{\frac{4}{3}} \cdot \omega^2\Big) \\ &= \frac{4}{3}\|\Upsilon\|_\omega^{-\frac{4}{3}} \cdot d(\log\|\Upsilon\|_\omega) \wedge dr \wedge \mathrm{Im}\,(\Upsilon) \\ &\quad - \frac{1}{6}\|\Upsilon\|_\omega^{\frac{4}{3}} \cdot d(\log\|\Upsilon\|_\omega) \wedge \omega^2. \end{aligned} \tag{6.11}$$

Taking the Hodge star, we get

$$\begin{aligned}
\star_\varphi d\psi &= \frac{4}{3}\|\Upsilon\|_\omega^{-\frac{2}{3}} \cdot \star_\varphi \Big[d(\log\|\Upsilon\|_\omega) \wedge dr \wedge \mathrm{Im}\,(\Upsilon)\Big] \\
&\quad - \frac{1}{6}\|\Upsilon\|_\omega^{\frac{4}{3}} \cdot \star_\varphi \Big[d(\log\|\Upsilon\|_\omega) \wedge \omega^2\Big] \\
&= \frac{4}{3}\|\Upsilon\|_\omega^{-\frac{2}{3}} \cdot \big(d(\log\|\Upsilon\|_\omega)\big)^{\sharp_\varphi} \lrcorner \star_\varphi \Big[dr \wedge \mathrm{Im}\,(\Upsilon)\Big] \\
&\quad - \frac{1}{3}\|\Upsilon\|_\omega^{\frac{4}{3}} \cdot \big(d(\log\|\Upsilon\|_\omega)\big)^{\sharp_\varphi} \lrcorner \star_\varphi \Big[\frac{1}{2}\omega^2\Big] \\
&= \frac{4}{3}\|\Upsilon\|_\omega^{-\frac{2}{3}} \cdot \big(2\|\Upsilon\|_\omega^{-\frac{2}{3}} \cdot \nabla(\log\|\Upsilon\|_\omega)\big) \lrcorner \Big[-\frac{1}{2}\|\Upsilon\|_\omega^{\frac{2}{3}} \cdot \mathrm{Re}\,(\Upsilon)\Big] \\
&\quad - \frac{1}{3}\|\Upsilon\|_\omega^{\frac{4}{3}} \cdot \big(2\|\Upsilon\|_\omega^{-\frac{2}{3}} \cdot \nabla(\log\|\Upsilon\|_\omega)\big) \lrcorner \Big[4\|\Upsilon\|_\omega^{-\frac{4}{3}} \cdot dr \wedge \omega^2\Big] \\
&= -\frac{4}{3}\|\Upsilon\|_\omega^{-\frac{2}{3}} \cdot \big(\nabla(\log\|\Upsilon\|_\omega)\big) \lrcorner \Big[\mathrm{Re}\,(\Upsilon) + 2dr \wedge \omega\Big] \\
&= 2\big(\nabla(\log\|\Upsilon\|_\omega^{-\frac{2}{3}})\big) \lrcorner \Big[\mathrm{Re}\,(\Upsilon) + 2dr \wedge \omega\Big]. \qquad (6.12)
\end{aligned}$$

In the above, we used the warped product structure of the induced metric in order to determine the $\sharp_\varphi$ operator on $M = S^1 \times X$ in terms of the regular $\sharp$ operator on $X$

Using Cartan's Magic Formula and the closedness of $\omega$ and $\Upsilon$, we get that

$$\Delta_{d,\varphi}\varphi = 2\mathcal{L}_{\nabla(\|\Upsilon\|_\omega^{-\frac{2}{3}})}\Big(-\mathrm{Re}\,(\Upsilon) - 2dr \wedge \omega\Big). \qquad (6.13)$$

□

Using the intermediate expressions in the above computations, we can also compute the individual torsion forms of $\varphi$.

**Lemma 6.2.2.** *If $\varphi$ is the $G_2$ structure defined by* (6.3)*, then*

$$\tau_0 = \tau_1 = \tau_3 = 0, \quad \tau_2 = 2\big(\nabla(\log\|\Upsilon\|_\omega^{-\frac{2}{3}})\big) \lrcorner \Big[\mathrm{Re}\,(\Upsilon) + 2dr \wedge \omega\Big]. \qquad (6.14)$$

If we assume that the Laplacian flow preserves our Ansatz, the result of Lemma 6.2.1 leads to the evolution equation

$$\frac{\partial}{\partial t}\Big(\mathrm{Re}\,(\Upsilon) - dr \wedge \omega\Big) = 2\mathcal{L}_{\nabla(\|\Upsilon\|_\omega^{-\frac{2}{3}})}\Big(-\mathrm{Re}\,(\Upsilon) - 2dr \wedge \omega\Big). \qquad (6.15)$$

Motivated by this equation, we consider Ansätze $(\omega, \Upsilon)$ on $X$ that satisfy the coupled differential equations

$$\frac{\partial}{\partial t}\omega = 4\mathcal{L}_{\nabla(\|\Upsilon\|_\omega^{-\frac{2}{3}})}\omega, \tag{6.16}$$

$$\frac{\partial}{\partial t}\Upsilon = -2\mathcal{L}_{\nabla(\|\Upsilon\|_\omega^{-\frac{2}{3}})}\Upsilon. \tag{6.17}$$

We note that if the evolving pair $(\omega, \Upsilon)$ satisfy the above equations, then they also satisfy (6.15).

**Remark 6.2.3.** We also note that the metric $g$ along the flow can be determined from the pair $(\omega, \Upsilon)$. From $\Upsilon$, we can obtain a complex structure $J$ by defining the subbundle $T^{1,0}X \subseteq T_{\mathbb{C}}X$ to be the kernel of $\Upsilon$, and then setting $J = \sqrt{-1}$ on $T^{1,0}X$ and $-\sqrt{-1}$ on its conjugate $T^{0,1}X$. From this, we can define $g(Y, Z) = \omega(JY, Z)$.

**Remark 6.2.4.** At this point, it is not a priori known if the structure $(\omega, \Upsilon)$ along the flow will remain compatible and integrable for all time. However, the solutions that will be presented in the sequel will satisfy any compatibility conditions required as they are obtained by pulling back compatible structures via diffeomorphisms.

In the following section, we will construct a solution $(\omega, \Upsilon)$ compatible with an integrable complex structure $J$ satisfying (6.16) and (6.17). To motivate the solution, we take a closer look at the Lie derivative in the first of the coupled equations. We have the identity

$$\mathcal{L}_{\nabla(\|\Upsilon\|_\omega^{-\frac{2}{3}})}\omega = 2\sqrt{-1}\partial\overline{\partial}(\|\Upsilon\|_\omega^{-\frac{2}{3}}), \tag{6.18}$$

which holds on any Kähler Calabi–Yau manifold. To show this, we first note that in local complex coordinates, we have

$$\nabla(\|\Upsilon\|_\omega^{-\frac{2}{3}}) = \frac{\partial}{\partial z^j}(\|\Upsilon\|_\omega^{-\frac{2}{3}})g^{j\overline{k}}\frac{\partial}{\partial \overline{z}^{\overline{k}}} + \frac{\partial}{\partial \overline{z}^k}(\|\Upsilon\|_\omega^{-\frac{2}{3}})g^{j\overline{k}}\frac{\partial}{\partial z^j}. \tag{6.19}$$

Using our local expressions for $\omega$ and Cartan's Magic Formula, we see that

$$\begin{aligned}
&\mathcal{L}_{\nabla(\|\Upsilon\|_{\omega}^{-\frac{2}{3}})}\omega \\
&= d\Big[\nabla(\|\Upsilon\|_{\omega}^{-\frac{2}{3}}) \lrcorner\, \omega\Big] \\
&= d\Big[\Big(\frac{\partial}{\partial z^j}(\|\Upsilon\|_{\omega}^{-\frac{2}{3}})g^{j\bar{k}}\frac{\partial}{\partial \bar{z}^{\bar{k}}} + \frac{\partial}{\partial \bar{z}^k}(\|\Upsilon\|_{\omega}^{-\frac{2}{3}})g^{j\bar{k}}\frac{\partial}{\partial z^j}\Big) \lrcorner \Big(\sqrt{-1}g_{p\bar{q}}dz^p \wedge d\bar{z}^{\bar{q}}\Big)\Big] \\
&= -\sqrt{-1}d\Big[\frac{\partial}{\partial z^j}(\|\Upsilon\|_{\omega}^{-\frac{2}{3}})dz^j\Big] + \sqrt{-1}d\Big[\frac{\partial}{\partial \bar{z}^k}(\|\Upsilon\|_{\omega}^{-\frac{2}{3}})d\bar{z}^{\bar{k}}\Big] \\
&= 2\sqrt{-1}\partial\bar{\partial}(\|\Upsilon\|_{\omega}^{-\frac{2}{3}}).
\end{aligned} \tag{6.20}$$

This identity indicates that the flow of $\omega$ is related to the MA$^{\frac{1}{3}}$ flow; this will be made precise in the following section.

A similar consideration of the other Lie derivative term tells us that the complex structure $J$ on $X$ must be changing in time. Indeed, we can check that

$$\begin{aligned}
&\mathcal{L}_{\nabla(\|\Upsilon\|_{\omega}^{-\frac{2}{3}})}\Upsilon \\
&= d\Big[\nabla(\|\Upsilon\|_{\omega}^{-\frac{2}{3}}) \lrcorner\, \Upsilon\Big] \\
&= d\Big[\Big(\frac{\partial}{\partial z^j}(\|\Upsilon\|_{\omega}^{-\frac{2}{3}})g^{j\bar{k}}\frac{\partial}{\partial \bar{z}^{\bar{k}}} + \frac{\partial}{\partial \bar{z}^k}(\|\Upsilon\|_{\omega}^{-\frac{2}{3}})g^{j\bar{k}}\frac{\partial}{\partial z^j}\Big) \lrcorner \Big(f dz^1 \wedge dz^2 \wedge dz^3\Big)\Big] \\
&= d\Big[f\frac{\partial}{\partial \bar{z}^k}(\|\Upsilon\|_{\omega}^{-\frac{2}{3}})g^{1\bar{k}}dz^2 \wedge dz^3 + \text{cyclic permutations}\Big]
\end{aligned} \tag{6.21}$$

and so in general, we get terms of type $(2,1)$ with respect to the current complex structure $J$. Hence in order for $\Omega$ to remain a $(3,0)$-form as it evolves so that we can define a moving $G_2$-structure

$$\varphi = \mathrm{Re}\,(\Upsilon) - dr \wedge \omega \tag{6.22}$$

on $M = S^1 \times X$, the complex structure must change as well. To solve the coupled system (6.16) - (6.17), we will act on compatible structures by a moving family of diffeomorphisms $\Theta$, so that from this point of view, the complex structure is fixed; this idea can be found in *e.g.,* [FPPZ21].

## 6.3 A Solution from the MA$^{\frac{1}{3}}$ Flow

Recall that $X$ is a Kähler Calabi–Yau threefold with Kähler form $\omega$ and nowhere-vanishing holomorphic $(3,0)$-form $\Upsilon$. Let $u$ be a smooth solution to the MA$^{\frac{1}{3}}$ flow

$$\frac{\partial}{\partial t}u = 12\Big(e^{-2\log\|\Upsilon\|_\omega}\cdot\frac{\det(\omega+\sqrt{-1}\partial\overline{\partial}u)}{\det\omega}\Big)^{\frac{1}{3}},\quad \omega+\sqrt{-1}\partial\overline{\partial}u>0,\quad (6.23)$$

on $X$ with $u_0 = 0$ (see §8 or [PZ20] for more details). From the flow, we may define a family of Kähler metrics $\widetilde{\omega} = \omega + \sqrt{-1}\partial\overline{\partial}u$ on $X$ for all time. The complex structure $J$ is fixed along this flow, and hence we have a family of Kähler triples $(\widetilde{\omega}, J, \widetilde{g})$ on $X$.

Recall that we have the local expression

$$\|\Upsilon\|^2_\omega = \frac{|f|^2}{\det g_{p\overline{q}}}. \quad (6.24)$$

Using this and the evolution equation (6.23), we can compute that

$$\begin{aligned}\frac{\partial}{\partial t}u &= 12\Big(\frac{\det(\omega+\sqrt{-1}\partial\overline{\partial}u)}{\|\Upsilon\|^2_\omega\cdot\det\omega}\Big)^{\frac{1}{3}}\\ &= 12\Big(\frac{\det(\omega+\sqrt{-1}\partial\overline{\partial}u)}{|f|^2}\Big)^{\frac{1}{3}}\\ &= 12(\|\Upsilon\|_{\widetilde{\omega}}^{-\frac{2}{3}}) \quad (6.25)\end{aligned}$$

and so

$$\frac{\partial}{\partial t}\widetilde{\omega} = \frac{\partial}{\partial t}\big(\omega+\sqrt{-1}\partial\overline{\partial}u\big) = \sqrt{-1}\partial\overline{\partial}\Big(\frac{\partial}{\partial t}u\Big) = 12\sqrt{-1}\partial\overline{\partial}(\|\Upsilon\|_{\widetilde{\omega}}^{-\frac{2}{3}}). \quad (6.26)$$

Using $\Upsilon$ and the smooth solution $\widetilde{\omega}$, we can define a time-dependent vector field

$$Y = -2\widetilde{\nabla}(\|\Upsilon\|_{\widetilde{\omega}}^{-\frac{2}{3}}). \quad (6.27)$$

Let $\Theta$ be the 1-parameter family of diffeomorphisms generated by $Y$ in the sense that

$$\frac{\partial}{\partial t}\Theta(p) = Y\big(\Theta(p)\big),\quad \Theta_0 = \mathrm{Id}_X. \quad (6.28)$$

This family exists for all time $t$ (see *e.g.,* Lemma 3.15 in [CK04]).

We can pull back our tensors of interest by this family of diffeomorphisms. Define

$$\widehat{\omega} = \Theta^*\widetilde{\omega},\quad \widehat{\Upsilon} = \Theta^*\Upsilon. \quad (6.29)$$

In general, $\widehat{\Upsilon}$ will not remain a $(3,0)$-form with respect to the original complex structure $J$ on $X$. However, by pulling $J$ back by the same family of diffeomorphisms $\Theta$, we get a flow of complex structures $\widehat{J} = \Theta^* J$ which keeps $\widehat{\Upsilon}$ a $(3,0)$-form. Further, since each of our tensors were obtained by pullback, we also get a Riemannian metric $\widehat{g} = \Theta^* \widetilde{g}$ compatible with both $\widehat{\omega}$ and $\widehat{J}$. That is, we have defined yet another family of Kähler triples $(\widehat{\omega}, \widehat{J}, \widehat{g})$ on $X$.

Using a computational identity in DeTurck's Trick and other tensorial properties, we will show that the pair $(\widehat{\omega}, \widehat{\Upsilon})$ is a solution to the coupled equations (6.16) - (6.17). Recall that the complex structure $\widehat{J}$ and in turn metric $\widehat{g}$ are determined by the pair $(\widehat{\omega}, \widehat{\Upsilon})$ and satisfy Kähler compatibility conditions described above.

We compute

$$
\begin{aligned}
\frac{\partial}{\partial t}\widehat{\omega} &= \frac{\partial}{\partial t}\big(\Theta^* \widetilde{\omega}\big) \\
&= \Theta^*\big(\mathcal{L}_Y \widetilde{\omega}\big) + \Theta^*\Big(\frac{\partial}{\partial t}\widetilde{\omega}\Big) \\
&= \mathcal{L}_{(\Theta^{-1})_* Y}\big(\Theta^* \widetilde{\omega}\big) + \Theta^*\big(12\sqrt{-1}\partial\overline{\partial}(\|\Upsilon\|_{\widetilde{\omega}}^{-\frac{2}{3}})\big) \\
&= \mathcal{L}_{-2(\Theta^{-1})_*[\widetilde{\nabla}(\|\Upsilon\|_{\widetilde{\omega}}^{-\frac{2}{3}})]}\widehat{\omega} + 12\sqrt{-1}\widehat{\partial}\widehat{\overline{\partial}}\big(\Theta^*(\|\Upsilon\|_{\widetilde{\omega}}^{-\frac{2}{3}})\big) \\
&= -2\mathcal{L}_{\widehat{\nabla}(\|\widehat{\Upsilon}\|_{\widehat{\omega}}^{-\frac{2}{3}})}\widehat{\omega} + 12\sqrt{-1}\widehat{\partial}\widehat{\overline{\partial}}(\|\widehat{\Upsilon}\|_{\widehat{\omega}}^{-\frac{2}{3}}).
\end{aligned} \tag{6.30}
$$

Using our expression for the Lie derivative (6.20), we see that

$$
\frac{\partial}{\partial t}\widehat{\omega} = 4\mathcal{L}_{\widehat{\nabla}(\|\widehat{\Upsilon}\|_{\widehat{\omega}}^{-\frac{2}{3}})}\widehat{\omega}, \tag{6.31}
$$

which is the first of our coupled equations.

A similar computation shows that

$$
\begin{aligned}
\frac{\partial}{\partial t}\widehat{\Upsilon} &= \frac{\partial}{\partial t}\big(\Theta^* \widetilde{\omega}\big) = \Theta^*\big(\mathcal{L}_Y \widetilde{\omega}\big) \\
&= \mathcal{L}_{(\Theta^{-1})_* Y}\big(\Theta^* \widetilde{\Upsilon}\big) \\
&= \mathcal{L}_{-2(\Theta^{-1})_*[\widetilde{\nabla}(\|\Upsilon\|_{\widetilde{\omega}}^{-\frac{2}{3}})]}\widehat{\Upsilon} \\
&= -2\mathcal{L}_{\widehat{\nabla}(\|\widehat{\Upsilon}\|_{\widehat{\omega}}^{-\frac{2}{3}})}\widehat{\omega},
\end{aligned} \tag{6.32}
$$

which is the second equation.

Combining what we have thus far, we have the following result:

**Theorem 6.3.1.** *Let $X$ be a Kähler Calabi–Yau threefold with Kähler form $\omega$ and nowhere-vanishing holomorphic $(3,0)$-form $\Upsilon$. Suppose we start the $G_2$-Laplacian coflow (7.1) on $M = S^1 \times X$ with initial data of the form*

$$\varphi_0 = \mathrm{Re}\,(\Upsilon) - dr \wedge \omega, \tag{6.33}$$

*then a solution to the flow exists for all time $t$ and is of the form*

$$\widehat{\varphi} = \mathrm{Re}\,(\widehat{\Upsilon}) - dr \wedge \widehat{\omega} \tag{6.34}$$

*where $\widehat{\omega} = \Theta^* \widetilde{\omega}$, $\widehat{\Upsilon} = \Theta^* \Upsilon$ with $\Theta$ being the 1-parameter family of diffeomorphisms generated by the (time-dependent) vector field*

$$Y = -2\widetilde{\nabla}(\|\Upsilon\|_{\widetilde{\omega}}^{-\frac{2}{3}}), \tag{6.35}$$

*and $\widetilde{\omega}$ solves the $MA^{\frac{1}{3}}$ flow on $X$ with initial condition $\omega_0 = \omega$.*

By uniqueness of the flow, we conclude that the Laplacian flow preserves the Ansatz (6.3) and is equivalent to the $\mathrm{MA}^{\frac{1}{3}}$ flow for this class of initial data.

# Chapter 7

# The (Modified) Laplacian Coflow

We now apply our previous methods to the (modified) Laplacian coflow. Here we shall also consider the case of non-trivial fibrations using contact Calabi–Yau 7-folds.

## 7.1 Properties of the Coflows

### 7.1.1 The Laplacian Coflow

In analogy with the Laplacian coflow, Karigiannis–McKay–Tsui [KMT12] considered a dual coflow on the 4-form $\psi$. This flow has evolution equation given by

$$\frac{\partial}{\partial t}\psi = \Delta_{d,\varphi}\psi, \tag{7.1}$$

where $\Delta_{d,\varphi} = dd^*_\varphi + d^*_\varphi d$ is again the Hodge Laplacian. One can check that (see *e.g.,* [Gri13]) inducing a flow on the dual 4-form induces a corresponding flow on its $G_2$-structure $\varphi$.

**Remark 7.1.1.** We note that the Laplacian coflow was originally introduced with a minus sign on the RHS of (7.1) by analogy with the heat equation. For this thesis, we instead follow the convention in [Lot20]. Additionally, the 4-form does not itself determine an orientation, however we may assume an initial orientation which will remain fixed along the flow.

Like the Laplacian flow, it is common to consider its restriction to coclosed $G_2$-structures. In this case, it reduces to

$$\frac{\partial}{\partial t}\psi = dd^*_\varphi\psi, \tag{7.2}$$

and so the flow preserves the coclosed condition given an initial coclosed $G_2$-structure $\psi$. Unlike the Laplacian flow, the short-time existence and uniqueness of the coflow even under the coclosed restriction is still unknown.

The stationary points of the coflow are torsion-free in both the compact and non-compact case (see [DGK21]).

### 7.1.2 The Modified Laplacian Coflow

Part of the reason why short-time existence and uniquenss of the Laplacian coflow is still unknown is because the Bryant–Xu method [BX11] does not apply. This occurs due to the presence of an additional negative term from $\Delta_d\psi$.

To remedy this, Grigorian [Gri13] proposed a modification of the coflow which adds a corresponding term to cancel the negative one. This flow has evolution equation given by

$$\frac{\partial}{\partial t}\psi = \Delta_{d,\varphi}\psi - 2d\Big[\Big(\frac{7}{4}\tau_0 - A\Big)\varphi\Big], \tag{7.3}$$

where $A \in \mathbb{R}$ is a constant. The addition of this new term makes the flow amenable to the Bryant–Xu method, and so the modified coflow has short-time existence and uniqueness. Unfortunately, in gaining these properties, we also introduce new stationary points for the flow.

**Example 7.1.2.** Consider a connected manifold $M$ with a nearly parallel $G_2$-structure with $d\varphi = \lambda\psi$ where $\lambda > 0$ is a constant. In this case, we have

$$\tau_0 = \lambda, \quad \tau_1 = \tau_2 = \tau_3 = 0. \tag{7.4}$$

Set $A = \frac{5}{4}\lambda$. We then compute that

$$\begin{aligned} \Delta_d\psi - 2d\Big[\Big(\frac{7}{4}\tau_0 - A\Big)\varphi\Big] &= d\star d\star\psi - d(\lambda\varphi) \\ &= d\star d\varphi - \lambda^2\psi \\ &= d\star(\lambda\psi) - \lambda^2\psi \\ &= d(\lambda\varphi) - \lambda^2\psi = 0, \end{aligned} \tag{7.5}$$

and so $\varphi$ is a fixed point for the modified coflow.

To consider both flows simultaneously, we use the evolution equation

$$\frac{\partial}{\partial t}\psi = \Delta_{d,\varphi}\psi + (C-2)d\Big[\Big(\frac{7}{4}\tau_0 - A\Big)\varphi\Big], \tag{7.6}$$

for a constant $C \in \mathbb{R}$. Here, the $C = 2$ case corresponds to the original Laplacian coflow, and $C = 0$ corresponds to the modified coflow.

**Remark 7.1.3.** By slightly augmenting the Bryant–Xu–Grigorian argument (as noted in [Gri13]), one can actually show that the flow has short-time existence and uniqueness for all $C < 1$.

**Remark 7.1.4.** One can actually check that the $G_2$-Anomaly flow as defined in (5.27) is in some sense a conformally coclosed version of the flow (7.6) with $C = \frac{4}{3}$ and $A = 0$. Thus in some sense, a better understanding of the (modified) coflow should provide insight into the $G_2$-Anomaly flow.

## 7.2 Evolution Equations for the Base Structures

We now apply the flow to our coclosed Ansätze on both a trivial fibration $S^1 \times X$ and on a contact Calabi–Yau manifold.

### 7.2.1 Trivial $S^1$-Fibrations

Recall that our structures on $M = S^1 \times X$ in this case are

$$\varphi = \mathrm{Re}\Big(\frac{1}{\|\Upsilon\|_\omega}\Upsilon\Big) - \|\Upsilon\|_\omega \cdot dr \wedge \omega, \tag{7.7}$$

$$g_\varphi = 4\|\Upsilon\|_\omega^2 \cdot dr \otimes dr + \frac{1}{2}g, \tag{7.8}$$

$$\mathrm{vol}_\varphi = \frac{1}{4}\|\Upsilon\|_\omega \cdot dr \wedge \mathrm{vol}, \tag{7.9}$$

$$\psi = -2dr \wedge \mathrm{Im}\,(\Upsilon) - \frac{1}{8}\omega^2. \tag{7.10}$$

Here, we also have the Hodge star relations

$$\star_\varphi \beta = (-1)^k 2^{(-2+k)}\|\Upsilon\|_\omega \cdot dr \wedge (\star\beta), \tag{7.11}$$

$$\star_\varphi(dr \wedge \beta) = 2^{(-4+k)}\frac{1}{\|\Upsilon\|_\omega} \cdot (\star\beta) \tag{7.12}$$

where $\beta \in \Omega^k(X)$.

We have the following result on the Hodge Laplacian of $\psi$:

**Lemma 7.2.1.** *If $\varphi$ is the $G_2$-structure defined by (7.7), then*

$$\Delta_{d,\varphi}\psi = 2\mathcal{L}_{\nabla(\log\|\Upsilon\|_\omega)}\Big( - 2dr \wedge \mathrm{Im}\,(\Upsilon) + \frac{1}{8}\omega^2\Big). \tag{7.13}$$

*Proof.* The proof is similar to that of Lemma 6.2.1. Since $\psi$ is a closed 4-form on a 7-manifold, we have

$$\Delta_{d,\varphi}\psi = dd^*_\varphi\psi = d \star_\varphi d\varphi. \tag{7.14}$$

Direct computations show that

$$\begin{aligned} d\varphi &= d\Big(\mathrm{Re}\,\Big(\frac{1}{\|\Upsilon\|_\omega}\Upsilon\Big) - \|\Upsilon\|_\omega \cdot dr \wedge \omega\Big) \\ &= -\frac{1}{\|\Upsilon\|_\omega} \cdot d(\log\|\Upsilon\|_\omega) \wedge \mathrm{Re}\,(\Upsilon) - \|\Upsilon\|_\omega \cdot d(\log\|\Upsilon\|_\omega) \wedge dr \wedge \omega. \end{aligned} \tag{7.15}$$

Taking the Hodge star of the above, we see that

$$\begin{aligned} \star_\varphi d\varphi &= -\frac{1}{\|\Upsilon\|_\omega} \cdot \star_\varphi\Big[d(\log\|\Upsilon\|_\omega) \wedge \mathrm{Re}\,(\Upsilon)\Big] \\ &\quad - \|\Upsilon\|_\omega \cdot \star_\varphi\Big[d(\log\|\Upsilon\|_\omega) \wedge dr \wedge \omega\Big] \\ &= \frac{1}{\|\Upsilon\|_\omega} \cdot \big(d(\log\|\Upsilon\|_\omega)\big)^{\sharp_\varphi} \lrcorner \star_\varphi \Big[\mathrm{Re}\,(\Upsilon)\Big] \\ &\quad + \|\Upsilon\|_\omega \cdot \big(d(\log\|\Upsilon\|_\omega)\big)^{\sharp_\varphi} \star_\varphi \Big[dr \wedge \omega\Big] \\ &= \frac{1}{\|\Upsilon\|_\omega} \cdot \big(2\nabla(\log\|\Upsilon\|_\omega)\big) \lrcorner \Big[ - 2\|\Upsilon\|_\omega \cdot dr \wedge \mathrm{Im}\,(\Upsilon)\Big] \\ &\quad + \|\Upsilon\|_\omega \cdot \big(2\nabla(\log\|\Upsilon\|_\omega)\big) \lrcorner \Big[\frac{1}{8}\frac{1}{\|\Upsilon\|_\omega} \cdot \omega^2\Big] \\ &= 2\big(\nabla(\log\|\Upsilon\|_\omega)\big) \lrcorner \Big[ - 2dr \wedge \mathrm{Im}\,(\Upsilon) + \frac{1}{8}\omega^2\Big]. \end{aligned} \tag{7.16}$$

Cartan's Magic Formula and the closedness of $\omega$ and $\Upsilon$ then give

$$\Delta_{d,\varphi}\psi = 2\mathcal{L}_{\nabla(\log\|\Upsilon\|_\omega)}\Big( - 2dr \wedge \mathrm{Im}\,(\Upsilon) + \frac{1}{8}\omega^2\Big) \tag{7.17}$$

as desired. □

We can also check the torsion forms of this structure.

**Lemma 7.2.2.** *If $\varphi$ is the $G_2$-structure defined by* (7.7)*, then*

$$\tau_0 = \tau_1 = \tau_2 = 0, \quad \tau_3 = 2\big(\nabla(\log\|\Upsilon\|_\omega)\big)\lrcorner\Big[-2dr\wedge \mathrm{Im}\,(\Upsilon) + \frac{1}{8}\omega^2\Big]. \tag{7.18}$$

*Proof.* We use the identities from §5.1 to compute the torsion forms of $\varphi$. Since $\psi$ is closed, we must have the $\tau_1 = \tau_2 = 0$.

Next, using the expression from above, we can see that

$$\begin{aligned}
&\varphi\wedge d\varphi\\
&= \Big[\mathrm{Re}\Big(\frac{1}{\|\Upsilon\|_\omega}\Upsilon\Big) - \|\Upsilon\|_\omega\cdot dr\wedge\omega\Big]\\
&\quad\wedge\Big[-\frac{1}{\|\Upsilon\|_\omega}\cdot d(\log\|\Upsilon\|_\omega)\wedge\mathrm{Re}\,(\Upsilon) - \|\Upsilon\|_\omega\cdot d(\log\|\Upsilon\|_\omega)\wedge dr\wedge\omega\Big].
\end{aligned} \tag{7.19}$$

This vanishes since $\mathrm{Re}\,(\Upsilon)\wedge\omega = 0$ due to type considerations and $\mathrm{Re}\,(\Upsilon)\wedge \mathrm{Re}\,(\Upsilon) = 0$ since $\mathrm{Re}\,(\Upsilon)$ is a 3-form.

From (5.19), we see that $\tau_0 = 0$. We then have

$$\tau_3 = \star_\varphi d\varphi = 2\big(\nabla(\log\|\Upsilon\|_\omega)\big)\lrcorner\Big[-2dr\wedge\mathrm{Im}\,(\Upsilon) + \frac{1}{8}\omega^2\Big]. \tag{7.20}$$

□

We again apply the flow (7.6) to our Ansatz. Using the previous two lemmata, we can write

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big(-2dr\wedge\mathrm{Im}\,(\Upsilon) - \frac{1}{8}\omega^2\Big)\\
&= 2\mathcal{L}_{\nabla(\log\|\Upsilon\|_\omega)}\Big(-2dr\wedge\mathrm{Im}\,(\Upsilon) + \frac{1}{8}\omega^2\Big)\\
&\quad + A(C-2)\frac{1}{\|\Upsilon\|_\omega}\cdot d(\log\|\Upsilon\|_\omega)\wedge\mathrm{Re}\,(\Upsilon)\\
&\quad + A(C-2)\|\Upsilon\|_\omega\cdot d(\log\|\Upsilon\|_\omega)\wedge dr\wedge\omega.
\end{aligned} \tag{7.21}$$

By matching terms, we can deduce the following:

**Theorem 7.2.3.** *Let $X$ be a Kähler Calabi–Yau threefold with Kähler form $\omega$ and nowhere-vanishing holomorphic $(3,0)$-form $\Upsilon$. Suppose we have a family of compatible* $\mathrm{SU}(3)$*-structures $(\widehat{\omega}, \widehat{\Upsilon})$ with initial conditions $\widehat{\omega}_0 = \omega$, $\widehat{\Upsilon}_0 = \Upsilon$ satisfying the coupled differential equations*

$$\frac{\partial}{\partial t}\widehat{\omega} = -2\mathcal{L}_{\widehat{\nabla}(\log \|\widehat{\Upsilon}\|_{\widehat{\omega}})}\widehat{\omega} + \beta, \tag{7.22}$$

$$\frac{\partial}{\partial t}\widehat{\Upsilon} = 2\mathcal{L}_{\widehat{\nabla}(\log \|\widehat{\Upsilon}\|_{\widehat{\omega}})}\widehat{\Upsilon} + \gamma, \tag{7.23}$$

*for some $\beta \in \Omega^2(X)$ and $\gamma \in \Omega^3(X)$. Let $\widehat{\varphi}$ be the family of $G_2$-structures on $M = S^1 \times X$ defined by*

$$\widehat{\varphi} = \mathrm{Re}\Big(\frac{1}{\|\widehat{\Upsilon}\|_{\widehat{\omega}}}\widehat{\Upsilon}\Big) - \|\widehat{\Upsilon}\|_{\widehat{\omega}} \cdot dr \wedge \widehat{\omega}, \tag{7.24}$$

*then $\widehat{\varphi}$ is a solution to the coflow* (7.6) *if and only if*

$$\beta \wedge \widehat{\omega} = -4A(C-2)\frac{1}{\|\widehat{\Upsilon}\|_{\widehat{\omega}}} \cdot d(\log \|\widehat{\Upsilon}\|_{\widehat{\omega}}) \wedge \mathrm{Re}\,(\widehat{\Upsilon}), \tag{7.25}$$

$$\mathrm{Im}\,(\gamma) = \frac{1}{2}A(C-2)\|\widehat{\Upsilon}\|_{\widehat{\omega}} \cdot d(\log \|\widehat{\Upsilon}\|_{\widehat{\omega}}) \wedge \widehat{\omega}. \tag{7.26}$$

*Proof.* We note that the dual 4-forms $\widehat{\psi}$ have the equation

$$\widehat{\psi} = -2dr \wedge \mathrm{Im}\,(\widehat{\Upsilon}) - \frac{1}{8}\widehat{\omega}^2. \tag{7.27}$$

Since the radial coordinate $r$ does not evolve, the 4-form changes by

$$\frac{\partial}{\partial t}\widehat{\psi} = -2dr \wedge \Big(\frac{\partial}{\partial t}\mathrm{Im}\,(\widehat{\Upsilon})\Big) - \frac{1}{8}\Big(\frac{\partial}{\partial t}\widehat{\omega}^2\Big). \tag{7.28}$$

The evolution equation for the flow (7.21) means that in order to solve the flow, we must have

$$\begin{aligned}
&-2dr \wedge \Big(\frac{\partial}{\partial t}\mathrm{Im}\,(\widehat{\Upsilon})\Big) - \frac{1}{8}\Big(\frac{\partial}{\partial t}\widehat{\omega}^2\Big) \\
&= 2\mathcal{L}_{\widehat{\nabla}(\log \|\widehat{\Upsilon}\|_{\widehat{\omega}})}\Big(-2dr \wedge \mathrm{Im}\,(\widehat{\Upsilon}) + \frac{1}{8}\widehat{\omega}^2\Big) \\
&\quad + A(C-2) \cdot \frac{1}{\|\widehat{\Upsilon}\|_{\widehat{\omega}}} \cdot d(\log \|\widehat{\Upsilon}\|_{\widehat{\omega}}) \wedge \mathrm{Re}\,(\widehat{\Upsilon}) \\
&\quad + A(C-2)\|\widehat{\Upsilon}\|_{\widehat{\omega}} \cdot d(\log \|\widehat{\Upsilon}\|_{\widehat{\omega}}) \wedge dr \wedge \widehat{\omega}.
\end{aligned} \tag{7.29}$$

Substituting in our assumptions (7.22) - (7.23), we are left with

$$\begin{aligned}
&- 2dr \wedge \mathrm{Im}\,(\gamma) - \frac{1}{4}\beta \wedge \widehat{\omega} \\
&= A(C-2)\frac{1}{\|\widehat{\Upsilon}\|_{\widehat{\omega}}} \cdot d(\log \|\widehat{\Upsilon}\|_{\widehat{\omega}}) \wedge \mathrm{Re}\,(\widehat{\Upsilon}) \\
&\quad + A(C-2)\|\widehat{\Upsilon}\|_{\widehat{\omega}} \cdot d(\log \|\widehat{\Upsilon}\|_{\widehat{\omega}}) \wedge dr \wedge \widehat{\omega}.
\end{aligned} \tag{7.30}$$

By interior multiplication by $\frac{\partial}{\partial r}$, we get

$$-2\mathrm{Im}\,(\gamma) = -A(C-2)\|\widehat{\Upsilon}\|_{\widehat{\omega}} \cdot d(\log \|\widehat{\Upsilon}\|_{\widehat{\omega}}) \wedge \widehat{\omega}. \tag{7.31}$$

Plugging this back into the equation then yields

$$-\frac{1}{4}\beta \wedge \widehat{\omega} = A(C-2)\frac{1}{\|\widehat{\Upsilon}\|_{\widehat{\omega}}} \cdot d(\log \|\widehat{\Upsilon}\|_{\widehat{\omega}}) \wedge \mathrm{Re}\,(\widehat{\Upsilon}), \tag{7.32}$$

as desired. □

From this, we can see that if the complex structure $J$ is fixed along the flow, then by type decomposition from (7.25) and (7.26), we must have

$$\beta^{(1,1)} = 0, \quad \mathrm{Im}\,(\gamma)^{(0,3)\oplus(3,0)} = 0. \tag{7.33}$$

**A Special Case**

In general, the conditions from the previous theorem are hard to solve. However, in the case of the original coflow ($C = 2$ or $A = 0$), we can make a simplification. The evolution equation becomes

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big( - 2dr \wedge \mathrm{Im}\,(\Upsilon) - \frac{1}{8}\omega^2\Big) \\
&= 2\mathcal{L}_{\nabla(\log \|\Upsilon\|_{\omega})}\Big( - 2dr \wedge \mathrm{Im}\,(\Upsilon) + \frac{1}{8}\omega^2\Big)
\end{aligned} \tag{7.34}$$

and so it is enough to solve the coupled system

$$\frac{\partial}{\partial t}\omega = -2\mathcal{L}_{\nabla(\log \|\Upsilon\|_{\omega})}\omega, \tag{7.35}$$

$$\frac{\partial}{\partial t}\Upsilon = 2\mathcal{L}_{\nabla(\log \|\Upsilon\|_{\omega})}\Upsilon. \tag{7.36}$$

A similar computation to (6.20) shows that

$$\mathcal{L}_{\nabla(\log\|\Upsilon\|_\omega)} = 2\sqrt{-1}\partial\overline{\partial}(\log\|\Upsilon\|_\omega). \tag{7.37}$$

Continuing this further using our local descriptions, we have

$$\begin{aligned} 2\sqrt{-1}\partial\overline{\partial}(\log\|\Upsilon\|_\omega) &= \sqrt{-1}\partial\overline{\partial}\Big(\log\frac{|f|^2}{\det g_{p\overline{q}}}\Big) \\ &= -\sqrt{-1}\partial\overline{\partial}(\log\det g_{p\overline{q}}) + \sqrt{-1}\partial\overline{\partial}(\log|f|^2) \\ &= \mathrm{Ric}(\omega, J), \end{aligned} \tag{7.38}$$

where the last term in the penultimate line vanishes since $f$ is a local holomorphic function. Combining these with the evolution equation for $\omega$, we obtain something reminiscent of the Kähler–Ricci flow.

We note that the other Lie derivative term involving $\Upsilon$ will again produce terms of type $(2,1)$ with respect to the complex structure $J$ on $X$. We thus conclude that the complex structure $J$ must move with respect to time and will use a similar pullback method to construct our solutions.

### A Solution from the Kähler–Ricci Flow

Let $\widetilde{\omega}$ be the unique smooth solution for the (rescaled) Kähler–Ricci flow

$$\frac{\partial}{\partial t}\widetilde{\omega} = -4\mathrm{Ric}(\widetilde{\omega}, J), \quad \widetilde{\omega}_0 = \omega \tag{7.39}$$

on $X$. Since $X$ is Kähler Calabi–Yau, we have that its first Chern class $c_1(X)$ vanishes and so this solution exists for all time $t \in [0,\infty)$ (see [Cao85]). As it did with the $\mathrm{MA}^{\frac{1}{3}}$ flow in §6.3, the complex structure $J$ remains fixed and we obtain a family of Kähler triples $(\widetilde{\omega}, J, \widetilde{g})$ on $X$.

We use the solutions $\widetilde{\omega}$ and the holomorphic $(3,0)$-form to define a time-dependent vector field

$$Y = 2\widetilde{\nabla}(\log\|\Upsilon\|_{\widetilde{\omega}}) \tag{7.40}$$

and let $\Theta$ be the 1-parameter family of diffeomorphisms that it generates.

Following §6.3, we pull back $\widetilde{\omega}$ and $\Upsilon$ by this family and define

$$\widehat{\omega} = \Theta^*\widetilde{\omega}, \quad \widehat{\Upsilon} = \Theta^*\Upsilon. \tag{7.41}$$

The same subtleties apply here, however this construction still satisfies the required compatibility conditions by generating another family of Kähler triples $(\widehat{\omega}, \widehat{J}, \widehat{g})$ on $X$, where $\widehat{J} = \Theta^* J$ and $\widehat{g} = \Theta^* \widetilde{g}$. In particular, our new 3-form $\widehat{\Upsilon}$ is a holomorphic $(3,0)$-form with respect to the moving complex structure $\widehat{J}$.

Applying the same DeTurck Trick computation here yields

$$\begin{aligned}
\frac{\partial}{\partial t}\widehat{\omega} &= \frac{\partial}{\partial t}\big(\Theta^* \widetilde{\omega}\big) \\
&= \Theta^*\big(\mathcal{L}_Y \widetilde{\omega}\big) + \Theta^*\Big(\frac{\partial}{\partial t}\widetilde{\omega}\Big) \\
&= \mathcal{L}_{(\Theta^{-1})_* Y}\big(\Theta^* \widetilde{\omega}\big) + \Theta^*\big( - 4\mathrm{Ric}(\widetilde{\omega}, J)\big) \\
&= \mathcal{L}_{2(\Theta^{-1})_*[\widetilde{\nabla}(\log \|\Upsilon\|_{\widetilde{\omega}})]}\widehat{\omega} - 4\mathrm{Ric}(\Theta^* \widetilde{\omega}, \Theta^* J) \\
&= 2\mathcal{L}_{\widehat{\nabla}(\log \|\widehat{\Upsilon}\|_{\widehat{\omega}})}\widehat{\omega} - 4\mathrm{Ric}(\widehat{\omega}, \widehat{J}) \\
&= -2\mathcal{L}_{\widehat{\nabla}(\log \|\widehat{\Upsilon}\|_{\widehat{\omega}})}\widehat{\omega}.
\end{aligned} \tag{7.42}$$

A similar calculation shows that

$$\frac{\partial}{\partial t}\Upsilon = 2\mathcal{L}_{\nabla(\log \|\Upsilon\|_{\omega})}\Upsilon. \tag{7.43}$$

and so the coupled system is satisfied. This shows the following:

**Theorem 7.2.4.** *Let $X$ be a Kähler Calabi–Yau threefold with Kähler form $\omega$ and nowhere-vanishing holomorphic $(3,0)$-form $\Upsilon$. Suppose we start the $G_2$-Laplacian flow (6.1) on $M = S^1 \times X$ with initial data of the form*

$$\varphi_0 = \mathrm{Re}\Big(\frac{1}{\|\Upsilon\|_{\omega}}\Upsilon\Big) - \|\Upsilon\|_{\omega} \cdot dr \wedge \omega, \tag{7.44}$$

*then a solution to the flow exists for all time $t$ and is of the form*

$$\widehat{\varphi} = \mathrm{Re}\Big(\frac{1}{\|\widehat{\Upsilon}\|_{\widehat{\omega}}}\widehat{\Upsilon}\Big) - \|\widehat{\Upsilon}\|_{\widehat{\omega}} \cdot dr \wedge \widehat{\omega} \tag{7.45}$$

*where $\widehat{\omega} = \Theta^* \widetilde{\omega}$, $\widehat{\Upsilon} = \Theta^* \Upsilon$ with $\Theta$ being the 1-parameter family of diffeomorphisms generated by the (time-dependent) vector field*

$$Y = 2\widetilde{\nabla}(\log \|\Upsilon\|_{\widetilde{\omega}}), \tag{7.46}$$

*and $\widetilde{\omega}$ solves the Kähler–Ricci flow on $X$ with initial condition $\omega_0 = \omega$.*

### 7.2.2 Contact Calabi–Yau 7-Folds

We return to the setting on a contact Calabi–Yau 7-fold $M$. Recall from §5.3.2 that our ansatz was of the form

$$\varphi = \mathrm{Re}\left(\frac{1}{\|\Upsilon\|_\omega}\Upsilon\right) - \|\Upsilon\|_\omega \cdot \eta \wedge \omega, \tag{7.47}$$

where $\eta$ is the contact form, $\omega = d\eta$ is the transverse Kähler form, and $\Upsilon$ is a nowhere-vanishing transverse $(3,0)$-form.

As we saw previously, this induces the following structures

$$g_\varphi = 4\|\Upsilon\|_\omega^2 \cdot \eta \otimes \eta + \frac{1}{2} g|_{\mathcal{D}}. \tag{7.48}$$

$$\mathrm{vol}_\varphi = \frac{1}{4}\|\Upsilon\|_\omega \cdot \eta \wedge \mathrm{vol}|_{\mathcal{D}}, \tag{7.49}$$

$$\psi = -2\eta \wedge \mathrm{Im}\,(\Upsilon) - \frac{1}{8}\omega^2, \tag{7.50}$$

and the identities

$$\star_\varphi \beta = (-1)^k 2^{(-2+k)} \|\Upsilon\|_\omega \cdot \eta \wedge (\star_B \beta), \tag{7.51}$$

$$\star_\varphi (\eta \wedge \beta) = 2^{(-4+k)} \frac{1}{\|\Upsilon\|_\omega} \cdot (\star_B \beta) \tag{7.52}$$

for $\beta \in \Omega_B^k(M)$.

We compute the Hodge Laplacian and torsion forms for such a $G_2$-structure. Even though the structures have similar expressions, we will see that the non-trivial topology will produce extra terms.

**Lemma 7.2.5.** *If $\varphi$ is the $G_2$-structure defined by* (7.47)*, then*

$$\begin{aligned}\Delta_{d,\varphi}\psi = {} & 2\mathcal{L}_{\nabla_{\mathcal{D}}(\log \|\Upsilon\|_\omega)}\Big( - 2\eta \wedge \mathrm{Im}\,(\Upsilon) + \frac{1}{8}\omega^2\Big) \\ & - 16\|\Upsilon\|_\omega^2 \cdot d(\log \|\Upsilon\|_\omega) \wedge \eta \wedge \omega - 8\|\Upsilon\|_\omega^2 \cdot \omega^2. \end{aligned} \tag{7.53}$$

*Proof.* Once again, we have

$$\Delta_{d,\varphi}\psi = d \star_\varphi d\varphi. \tag{7.54}$$

We compute that

$$d\varphi = d\Big(\mathrm{Re}\left(\frac{1}{\|\Upsilon\|_\omega}\Upsilon\right) - \|\Upsilon\|_\omega \cdot \eta \wedge \omega\Big)$$

$$= -\frac{1}{\|\Upsilon\|_\omega} \cdot d(\log \|\Upsilon\|_\omega) \wedge \mathrm{Re}\,(\Upsilon) \\ - \|\Upsilon\|_\omega \cdot d(\log \|\Upsilon\|_\omega) \wedge \eta \wedge \omega - \|\Upsilon\|_\omega \cdot \omega^2. \tag{7.55}$$

Taking the Hodge star of both sides, we obtain

$$\begin{aligned} \star_\varphi d\varphi &= -\frac{1}{\|\Upsilon\|_\omega} \cdot \star_\varphi \Big[ d(\log \|\Upsilon\|_\omega) \wedge \mathrm{Re}\,(\Upsilon) \Big] \\ &\quad - \|\Upsilon\|_\omega \cdot \star_\varphi \Big[ d(\log \|\Upsilon\|_\omega) \wedge \eta \wedge \omega \Big] - \|\Upsilon\|_\omega \cdot \star_\varphi \omega^2 \\ &= \frac{1}{\|\Upsilon\|_\omega} \cdot \big( d(\log \|\Upsilon\|_\omega) \big)^{\sharp_\varphi} \lrcorner \star_\varphi \Big[ \mathrm{Re}\,(\Upsilon) \Big] \\ &\quad + \|\Upsilon\|_\omega \cdot \big( d(\log \|\Upsilon\|_\omega) \big)^{\sharp_\varphi} \lrcorner \star_\varphi \Big[ \eta \wedge \omega \Big] - 8 \|\Upsilon\|_\omega^2 \cdot \eta \wedge \omega \\ &= \frac{1}{\|\Upsilon\|_\omega} \cdot \big( 2\nabla_{\mathcal{D}} (\log \|\Upsilon\|_\omega) \big) \lrcorner \Big[ -2 \|\Upsilon\|_\omega \cdot \eta \wedge \mathrm{Im}\,(\Upsilon) \Big] \\ &\quad + \|\Upsilon\|_\omega \cdot \big( 2\nabla_{\mathcal{D}} (\log \|\Upsilon\|_\omega) \big) \lrcorner \Big[ \frac{1}{8} \frac{1}{\|\Upsilon\|_\omega} \cdot \omega^2 \Big] - 8 \|\Upsilon\|_\omega^2 \cdot \eta \wedge \omega \\ &= 2 \big( \nabla_{\mathcal{D}} (\log \|\Upsilon\|_\omega) \big) \lrcorner \Big[ -2\eta \wedge \mathrm{Im}\,(\Upsilon) + \frac{1}{8} \omega^2 \Big] \\ &\quad - 8 \|\Upsilon\|_\omega^2 \cdot \eta \wedge \omega. \end{aligned} \tag{7.56}$$

Once again, using Cartan's Magic Formula, we see that

$$\begin{aligned} \Delta_{d,\varphi} \psi &= d \star_\varphi d\varphi \\ &= 2 \mathcal{L}_{\nabla_{\mathcal{D}} (\log \|\Upsilon\|_\omega)} \Big( -2\eta \wedge \mathrm{Im}\,(\Upsilon) + \frac{1}{8} \omega^2 \Big) \\ &\quad - 16 \|\Upsilon\|_\omega^2 \cdot d(\log \|\Upsilon\|_\omega) \wedge \eta \wedge \omega - 8 \|\Upsilon\|_\omega^2 \cdot \omega^2. \end{aligned} \tag{7.57}$$

□

**Lemma 7.2.6.** *If $\varphi$ is the $G_2$-structure defined by* (7.47)*, then*

$$\begin{aligned} \tau_1 &= \tau_2 = 0, \quad \tau_0 = \frac{24}{7} \|\Upsilon\|_\omega, \\ \tau_3 &= 2 \big( \nabla_{\mathcal{D}} (\log \|\Upsilon\|_\omega) \big) \lrcorner \Big[ -2\eta \wedge \mathrm{Im}\,(\Upsilon) + \frac{1}{8} \omega^2 \Big] \\ &\quad - \frac{24}{7} \mathrm{Re}\,(\Upsilon) - \frac{32}{7} \|\Upsilon\|^2 \cdot \eta \wedge \omega. \end{aligned} \tag{7.58}$$

*Proof.* Using (5.19), we have

$$\begin{aligned}\tau_0 &= \frac{1}{7}\star_\varphi(\varphi\wedge d\varphi)\\ &= \frac{1}{7}\star_\varphi\Big(\|\Upsilon\|_\omega^2\cdot\eta\wedge\omega^3\Big)\\ &= \frac{24}{7}\|\Upsilon\|_\omega\star_\varphi\Big(\frac{1}{4}\|\Upsilon\|_\omega\cdot\eta\wedge\mathrm{vol}|_{\mathcal{D}}\Big)\\ &= \frac{24}{7}\|\Upsilon\|_\omega. \end{aligned}\tag{7.59}$$

Since $\psi$ is closed, we get that

$$\begin{aligned}\tau_3 &= \star_\varphi d\varphi - \star(\tau_0\psi)\\ &= 2\big(\nabla_{\mathcal{D}}(\log\|\Upsilon\|_\omega)\big)\lrcorner\Big[-2\eta\wedge\mathrm{Im}\,(\Upsilon)+\frac{1}{8}\omega^2\Big]-8\|\Upsilon\|^2\cdot\eta\wedge\omega\\ &\quad-\frac{24}{7}\|\Upsilon\|_\omega\cdot\Big(\mathrm{Re}\Big(\frac{1}{\|\Upsilon\|_\omega}\Upsilon\Big)-\|\Upsilon\|_\omega\cdot\eta\wedge\omega\Big)\\ &= 2\big(\nabla_{\mathcal{D}}(\log\|\Upsilon\|_\omega)\big)\lrcorner\Big[-2\eta\wedge\mathrm{Im}\,(\Upsilon)+\frac{1}{8}\omega^2\Big]\\ &\quad-\frac{24}{7}\mathrm{Re}\,(\Upsilon)-\frac{32}{7}\|\Upsilon\|^2\cdot\eta\wedge\omega. \end{aligned}\tag{7.60}$$

□

We now apply the flow (7.6) to our $G_2$-structures. Using our previous two results, we get the evolution equation

$$\begin{aligned}&\frac{\partial}{\partial t}\Big(-2\eta\wedge\mathrm{Im}\,(\Upsilon)-\frac{1}{8}\omega^2\Big)\\ &= 2\mathcal{L}_{\nabla_{\mathcal{D}}(\log\|\Upsilon\|_\omega)}\Big(-2\eta\wedge\mathrm{Im}\,(\Upsilon)+\frac{1}{8}\omega^2\Big)\\ &\quad-16\|\Upsilon\|_\omega^2\cdot d(\log\|\Upsilon\|_\omega)\wedge\eta\wedge\omega-8\|\Upsilon\|_\omega^2\cdot\omega^2\\ &\quad+(C-2)d\Big[\Big(6\|\Upsilon\|_\omega-A\Big)\cdot\Big(\mathrm{Re}\Big(\frac{1}{\|\Upsilon\|_\omega}\Upsilon\Big)-\|\Upsilon\|_\omega\cdot\eta\wedge\omega\Big)\Big]\\ &= 2\mathcal{L}_{\nabla_{\mathcal{D}}(\log\|\Upsilon\|_\omega)}\Big(-2\eta\wedge\mathrm{Im}\,(\Upsilon)+\frac{1}{8}\omega^2\Big)\\ &\quad-16\|\Upsilon\|_\omega^2\cdot d(\log\|\Upsilon\|_\omega)\wedge\eta\wedge\omega-8\|\Upsilon\|_\omega^2\cdot\omega^2\\ &\quad-12(C-2)\|\Upsilon\|_\omega^2\cdot d(\log\|\Upsilon\|_\omega)\wedge\eta\wedge\omega-6(C-2)\|\Upsilon\|_\omega^2\cdot\omega^2\\ &\quad+A(C-2)\frac{1}{\|\Upsilon\|_\omega}\cdot d(\log\|\Upsilon\|_\omega)\wedge\mathrm{Re}\,(\Upsilon)\\ &\quad+A(C-2)\|\Upsilon\|_\omega\cdot d(\log\|\Upsilon\|_\omega)\wedge\eta\wedge\omega+A(C-2)\|\Upsilon\|_\omega\cdot\omega^2 \end{aligned}$$

$$
\begin{aligned}
&= 2\mathcal{L}_{\nabla_{\mathcal{D}}(\log\|\Upsilon\|_\omega)}\Big( -2\eta\wedge \mathrm{Im}\,(\Upsilon) + \frac{1}{8}\omega^2\Big) \\
&\quad + \Big[A(C-2) - 12\Big(C-\frac{2}{3}\Big)\|\Upsilon\|_\omega\Big]\cdot \|\Upsilon\|_\omega \cdot d(\log\|\Upsilon\|_\omega)\wedge\eta\wedge\omega \\
&\quad + \Big[A(C-2) - 6\Big(C-\frac{2}{3}\Big)\|\Upsilon\|_\omega\Big]\cdot \|\Upsilon\|_\omega \cdot \omega^2 \\
&\quad + A(C-2)\frac{1}{\|\Upsilon\|_\omega}\cdot d(\log\|\Upsilon\|_\omega)\wedge \mathrm{Re}\,(\Upsilon).
\end{aligned}
\tag{7.61}
$$

We first consider how our flow and ansatz are compatible with a particular deformation of type II (see Appendix C).

**Theorem 7.2.7.** *Let $M$ be a contact Calabi–Yau 7-fold with contact form $\eta$, transverse Kähler form $\omega = d\eta$, and nowhere-vanishing transverse holomorphic $(3,0)$-form $\Upsilon$. Suppose we have a family of contact forms $\widehat{\eta} = \eta + d^c h$ and compatible transverse* SU(3)*-structures $(\widehat{\omega} = d\widehat{\eta}, \widehat{\Upsilon})$ on $M$ with initial conditions $\widehat{\eta}_0 = \eta$, $\widehat{\omega}_0 = \omega$, $\widehat{\Upsilon}_0 = \Upsilon$ satisfying the coupled differential equations*

$$
\frac{\partial}{\partial t}\widehat{\omega} = -2\mathcal{L}_{\widehat{\nabla}_{\mathcal{D}}(\log\|\widehat{\Upsilon}\|_{\widehat{\omega}})}\widehat{\omega} + \beta, \tag{7.62}
$$

$$
\frac{\partial}{\partial t}\widehat{\Upsilon} = 2\mathcal{L}_{\widehat{\nabla}_{\mathcal{D}}(\log\|\widehat{\Upsilon}\|_{\widehat{\omega}})}\widehat{\Upsilon} + \gamma, \tag{7.63}
$$

*for some $h \in C^\infty_B(M)$, $\beta \in \Omega^2_B(M)$ and $\gamma \in \Omega^3_B(M)$. Let $\widehat{\varphi}$ be the family of $G_2$-structures on $M$ defined by*

$$
\widehat{\varphi} = \mathrm{Re}\,\Big(\frac{1}{\|\widehat{\Upsilon}\|_{\widehat{\omega}}}\widehat{\Upsilon}\Big) - \|\widehat{\Upsilon}\|_{\widehat{\omega}}\cdot dr\wedge\widehat{\omega}, \tag{7.64}
$$

*then $\widehat{\varphi}$ is a solution to the coflow* (7.6) *if and only if*

$$
\begin{aligned}
\beta\wedge\widehat{\omega} = &-4\Big[A(C-2) - 6\Big(C-\frac{2}{3}\Big)\|\widehat{\Upsilon}\|_{\widehat{\omega}}\Big]\cdot\|\widehat{\Upsilon}\|_{\widehat{\omega}}\cdot\widehat{\omega}^2 \\
&- 4A(C-2)\frac{1}{\|\widehat{\Upsilon}\|_{\widehat{\omega}}}\cdot d(\log\|\widehat{\Upsilon}\|_{\widehat{\omega}})\wedge\mathrm{Re}\,(\widehat{\Upsilon}) \\
&+ 16\Big(\big(\widehat{\nabla}_{\mathcal{D}}(\log\|\widehat{\Upsilon}\|_{\widehat{\omega}})\big)\lrcorner\,\widehat{\omega}\Big)\wedge\mathrm{Im}\,(\widehat{\Upsilon}) \\
&+ 16 d\Big(\widehat{\nabla}_{\mathcal{D}}(\log\|\widehat{\Upsilon}\|_{\widehat{\omega}})\big)\lrcorner\, d^c h\Big)\wedge\mathrm{Im}\,(\widehat{\Upsilon}),
\end{aligned}
\tag{7.65}
$$

$$
\mathrm{Im}\,(\gamma) = \frac{1}{2}\Big[A(C-2) - 12\Big(C-\frac{2}{3}\Big)\|\widehat{\Upsilon}\|_{\widehat{\omega}}\Big]\cdot\|\widehat{\Upsilon}\|_{\widehat{\omega}}\cdot d(\log\|\widehat{\Upsilon}\|_{\widehat{\omega}})\wedge\widehat{\omega}. \tag{7.66}
$$

*Proof.* The proof is similar to that of Theorem 7.2.3. Using that

$$\widehat{\psi} = -2\widehat{\eta} \wedge \mathrm{Im}\,(\widehat{\Upsilon}) - \frac{1}{8}\widehat{\omega}^2, \tag{7.67}$$

we get that

$$\frac{\partial}{\partial t}\widehat{\psi} = -2\widehat{\eta} \wedge \Big(\frac{\partial}{\partial t}\mathrm{Im}\,(\widehat{\Upsilon})\Big) - \frac{1}{8}\Big(\frac{\partial}{\partial t}\widehat{\omega}^2\Big) - 2\Big(\frac{\partial}{\partial t}\widehat{\eta}\Big) \wedge \mathrm{Im}\,(\Upsilon). \tag{7.68}$$

To satisfy the flow, we must then have

$$\begin{aligned}
&-2\widehat{\eta} \wedge \Big(\frac{\partial}{\partial t}\mathrm{Im}\,(\widehat{\Upsilon})\Big) - \frac{1}{8}\Big(\frac{\partial}{\partial t}\widehat{\omega}^2\Big) - 2\Big(\frac{\partial}{\partial t}\widehat{\eta}\Big) \wedge \mathrm{Im}\,(\widehat{\Upsilon}) \\
&= 2\mathcal{L}_{\widehat{\nabla}_{\mathcal{D}}(\log\|\widehat{\Upsilon}\|_{\widehat{\omega}})}\Big(-2\widehat{\eta} \wedge \mathrm{Im}\,(\widehat{\Upsilon}) + \frac{1}{8}\widehat{\omega}^2\Big) \\
&\quad + \Big[A(C-2) - 12\Big(C - \frac{2}{3}\Big)\|\widehat{\Upsilon}\|_{\widehat{\omega}}\Big] \cdot \|\widehat{\Upsilon}\|_{\widehat{\omega}} \cdot d(\log\|\widehat{\Upsilon}\|_{\widehat{\omega}}) \wedge \widehat{\eta} \wedge \widehat{\omega} \\
&\quad + \Big[A(C-2) - 6\Big(C - \frac{2}{3}\Big)\|\widehat{\Upsilon}\|_{\widehat{\omega}}\Big] \cdot \|\widehat{\Upsilon}\|_{\widehat{\omega}} \cdot \widehat{\omega}^2 \\
&\quad + A(C-2)\frac{1}{\|\widehat{\Upsilon}\|_{\widehat{\omega}}} \cdot d(\log\|\widehat{\Upsilon}\|_{\widehat{\omega}}) \wedge \mathrm{Re}\,(\widehat{\Upsilon}).
\end{aligned} \tag{7.69}$$

We first notice that

$$\begin{aligned}
&\Big(\mathcal{L}_{\widehat{\nabla}_{\mathcal{D}}(\log\|\widehat{\Upsilon}\|_{\widehat{\omega}})}\widehat{\eta}\Big) \wedge \mathrm{Im}\,(\widehat{\Upsilon}) \\
&= \Big(\big(\widehat{\nabla}_{\mathcal{D}}(\log\|\widehat{\Upsilon}\|_{\widehat{\omega}})\big) \lrcorner\, \widehat{\omega}\Big) \wedge \mathrm{Im}\,(\widehat{\Upsilon}) + d\Big(\widehat{\nabla}_{\mathcal{D}}(\log\|\widehat{\Upsilon}\|_{\widehat{\omega}})\big) \lrcorner\, d^c h\Big) \wedge \mathrm{Im}\,(\widehat{\Upsilon}),
\end{aligned}$$

which is a basic form.

If we plug in our assumptions into (7.69), we get

$$\begin{aligned}
&-2\widehat{\eta} \wedge \mathrm{Im}\,(\gamma) - \frac{1}{4}\beta \wedge \widehat{\omega} - 2\Big(\frac{\partial}{\partial t}d^c h\Big) \wedge \mathrm{Im}\,(\widehat{\Upsilon}) \\
&= -4\Big(\big(\widehat{\nabla}_{\mathcal{D}}(\log\|\widehat{\Upsilon}\|_{\widehat{\omega}})\big) \lrcorner\, \widehat{\omega}\Big) \wedge \mathrm{Im}\,(\widehat{\Upsilon}) \\
&\quad - 4d\Big(\widehat{\nabla}_{\mathcal{D}}(\log\|\widehat{\Upsilon}\|_{\widehat{\omega}})\big) \lrcorner\, d^c h\Big) \wedge \mathrm{Im}\,(\widehat{\Upsilon}) \\
&\quad + \Big[A(C-2) - 12\Big(C - \frac{2}{3}\Big)\|\widehat{\Upsilon}\|_{\widehat{\omega}}\Big] \cdot \|\widehat{\Upsilon}\|_{\widehat{\omega}} \cdot d(\log\|\widehat{\Upsilon}\|_{\widehat{\omega}}) \wedge \widehat{\eta} \wedge \widehat{\omega} \\
&\quad + \Big[A(C-2) - 6\Big(C - \frac{2}{3}\Big)\|\widehat{\Upsilon}\|_{\widehat{\omega}}\Big] \cdot \|\widehat{\Upsilon}\|_{\widehat{\omega}} \cdot \widehat{\omega}^2 \\
&\quad + A(C-2)\frac{1}{\|\widehat{\Upsilon}\|_{\widehat{\omega}}} \cdot d(\log\|\widehat{\Upsilon}\|_{\widehat{\omega}}) \wedge \mathrm{Re}\,(\widehat{\Upsilon}).
\end{aligned} \tag{7.70}$$

We recall that the Reeb vector field $\xi$ stays fixed under this deformation. Contracting with $\xi$, we get

$$
\begin{aligned}
&-2\mathrm{Im}\,(\gamma) \\
&= -\Big[A(C-2) - 12\Big(C - \frac{2}{3}\Big)\|\widehat{\Upsilon}\|_{\widehat{\omega}}\Big] \cdot \|\widehat{\Upsilon}\|_{\widehat{\omega}} \cdot d(\log\|\widehat{\Upsilon}\|_{\widehat{\omega}}) \wedge \widehat{\omega}
\end{aligned}
\tag{7.71}
$$

and hence

$$
\begin{aligned}
&-\frac{1}{4}\beta \wedge \widehat{\omega} - 2\Big(\frac{\partial}{\partial t} d^c h\Big) \wedge \mathrm{Im}\,(\Upsilon) \\
&= -4\Big(\big(\widehat{\nabla}_{\mathcal{D}}(\log\|\widehat{\Upsilon}\|_{\widehat{\omega}})\big) \lrcorner\, \widehat{\omega}\Big) \wedge \mathrm{Im}\,(\widehat{\Upsilon}) \\
&\quad - 4d\Big(\widehat{\nabla}_{\mathcal{D}}(\log\|\widehat{\Upsilon}\|_{\widehat{\omega}})\big) \lrcorner\, d^c h\Big) \wedge \mathrm{Im}\,(\widehat{\Upsilon}) \\
&\quad + \Big[A(C-2) - 6\Big(C - \frac{2}{3}\Big)\|\widehat{\Upsilon}\|_{\widehat{\omega}}\Big] \cdot \|\widehat{\Upsilon}\|_{\widehat{\omega}} \cdot \widehat{\omega}^2 \\
&\quad + A(C-2)\frac{1}{\|\widehat{\Upsilon}\|_{\widehat{\omega}}} \cdot d(\log\|\widehat{\Upsilon}\|_{\widehat{\omega}}) \wedge \mathrm{Re}\,(\widehat{\Upsilon}).
\end{aligned}
\tag{7.72}
$$

□

Since this deformation of type II fixes the transverse complex structure $J$, applying type decomposition to (7.65) and (7.66) implies that we again must have

$$
\mathrm{Im}\,(\gamma)^{(0,3)\oplus(3,0)} = 0. \tag{7.73}
$$

The additional terms, however, no longer yield a type condition on $\beta$.

**Remark 7.2.8.** As in the previous section, we notice that choosing $A = 0$ and $C = 2$ greatly simplifies the evolution equation to

$$
\begin{aligned}
&\frac{\partial}{\partial t}\Big(-2\eta \wedge \mathrm{Im}\,(\Upsilon) - \frac{1}{8}\omega^2\Big) \\
&= 2\mathcal{L}_{\nabla_{\mathcal{D}}(\log\|\Upsilon\|_{\omega})}\Big(-2\eta \wedge \mathrm{Im}\,(\Upsilon) + \frac{1}{8}\omega^2\Big).
\end{aligned}
\tag{7.74}
$$

This suggests that we may mimic the special case there (while also allowing $\eta$ to evolve with time) and write

$$
\frac{\partial}{\partial t}\eta = 2\mathcal{L}_{\nabla_{\mathcal{D}}(\log\|\Upsilon\|_{\omega})}\eta, \tag{7.75}
$$

$$
\frac{\partial}{\partial t}\omega = -2\mathcal{L}_{\nabla_{\mathcal{D}}(\log\|\Upsilon\|_{\omega})}\omega, \tag{7.76}
$$

$$\frac{\partial}{\partial t}\Upsilon = 2\mathcal{L}_{\nabla_{\mathcal{D}}(\log\|\Upsilon\|_{\omega})}\Upsilon. \tag{7.77}$$

While seeming promising as an Ansatz, we note that this is incompatible with the relation $\omega = d\eta$.

### Breaking the Sasakian Structure

As seen in the previous remark, the relation $\omega = d\eta$ becomes rather restrictive. We now consider a different type of deformation that allows for more degrees of freedom. In particular, we allow the transverse structure to vary within the basic cohomology class $[d\eta]_B$. Note that the transverse complex structure $J$ is still fixed by this additional freedom.

Let $\mathcal{S} = (\xi, \eta, \Phi, g)$ be a Sasakian structure on $M$ and $\omega' \in [d\eta]_B$. Since the transverse complex structure given remains the same there, we can treat $(\omega', \Upsilon)$ as a transverse SU(3)-structure in its own right. This can, in some sense, be considered a breaking of the Saskain structure, since the transverse Kähler form $\omega'$ is no longer determined by the contact form $\eta$. By El Kacimi-Alaoui's transverse $\sqrt{-1}\partial\overline{\partial}$-Lemma [EKA90] (see Appendix C), there exists a basic function $\zeta$ such that

$$\omega = d\eta + dd^c\zeta. \tag{7.78}$$

In a similar manner to §5.3, we define a $G_2$-structure using $(\omega', \Upsilon)$ by

$$\varphi = \operatorname{Re}\Big(\frac{1}{\|\Upsilon\|_{\omega'}}\Upsilon\Big) - \|\Upsilon\|_{\omega'}\cdot\eta\wedge\omega'. \tag{7.79}$$

In this case, we have the metric

$$g_{\varphi} = 4\|\Upsilon\|^2_{\omega'}\cdot\eta\otimes\eta + \frac{1}{2}g'|_{\mathcal{D}} \tag{7.80}$$

and volume form

$$\mathrm{vol}_{\varphi} = \frac{1}{4}\|\Upsilon\|_{\omega'}\cdot\eta\wedge\mathrm{vol}'|_{\mathcal{D}}. \tag{7.81}$$

Additionally if $\beta$ is a basic $k$-form, then the Hodge star acts by

$$\star_{\varphi}\beta = (-1)^k 2^{(-2+k)}\|\Upsilon\|_{\omega'}\cdot\eta\wedge(\star'_B\beta), \tag{7.82}$$

$$\star_{\varphi}(\eta\wedge\beta) = 2^{(-4+k)}\frac{1}{\|\Upsilon\|_{\omega'}}\cdot(\star'_B\beta), \tag{7.83}$$

and so the dual 4-form is given by

$$\psi = -2\eta \wedge \mathrm{Im}\,(\Upsilon) - \frac{1}{8}(\omega')^2, \tag{7.84}$$

and is hence closed.

Once again, we have the Hodge Laplacian and torsion forms associated to these structures.

**Lemma 7.2.9.** *If $\varphi$ is the $G_2$-structure defined in* (7.79)*, then*

$$\begin{aligned}\Delta_{d,\varphi}\psi = {} & 2\mathcal{L}_{\nabla'_{\mathcal{D}}(\log\|\Upsilon\|_{\omega'})}\Big( -2\eta\wedge\mathrm{Im}\,(\Upsilon) + \frac{1}{8}(\omega')^2\Big) \\ & - 24\|\Upsilon\|^2_{\omega'}\cdot d(\log\|\Upsilon\|_{\omega'})\wedge\eta\wedge\omega' \\ & + 8\|\Upsilon\|^2_{\omega'}\cdot d(\log\|\Upsilon\|_{\omega'})\wedge\eta\wedge d\eta \\ & - 12\|\Upsilon\|^2_{\omega'}\cdot d\eta\wedge\omega' + 4\|\Upsilon\|^2_{\omega'}\cdot d\eta\wedge d\eta.\end{aligned} \tag{7.85}$$

**Lemma 7.2.10.** *If $\varphi$ is the $G_2$-structure defined by* (7.79)*, then*

$$\begin{aligned}&\tau_1 = \tau_2 = 0, \quad \tau_0 = \frac{24}{7}\|\Upsilon\|_{\omega'}, \\ &\tau_3 = 2\big(\nabla'_{\mathcal{D}}(\log\|\Upsilon\|_{\omega'})\big)\lrcorner\Big[-2\eta\wedge\mathrm{Im}\,(\Upsilon) + \frac{1}{8}(\omega')^2\Big] \\ &\qquad - \frac{24}{7}\mathrm{Re}\,(\Upsilon) - \frac{60}{7}\|\Upsilon\|^2_{\omega'}\cdot\eta\wedge\omega' + 4\|\Upsilon\|^2_{\omega'}\cdot\eta\wedge d\eta.\end{aligned} \tag{7.86}$$

The proofs are similar to before but require the use of some transverse Kähler identities. Further, one can check that these coincide with our previous identities in the case where $\omega' = d\eta$.

Using our evolution equation, we get

$$\begin{aligned}&\frac{\partial}{\partial t}\Big( -2\eta\wedge\mathrm{Im}\,(\Upsilon) - \frac{1}{8}(\omega')^2\Big) \\ &= 2\mathcal{L}_{\nabla'_{\mathcal{D}}(\log\|\Upsilon\|_{\omega'})}\Big( -2\eta\wedge\mathrm{Im}\,(\Upsilon) + \frac{1}{8}(\omega')^2\Big) \\ &\quad + \Big[A(C-2) - 12C\|\Upsilon\|_{\omega'}\Big]\cdot\|\Upsilon\|_{\omega'}\cdot d(\log\|\Upsilon\|_{\omega'})\wedge\eta\wedge\omega' \\ &\quad + 8\|\Upsilon\|^2_{\omega'}\cdot d(\log\|\Upsilon\|_{\omega'})\wedge\eta\wedge d\eta \\ &\quad + \Big[A(C-2) - 6C\|\Upsilon\|_{\omega'}\Big]\cdot\|\Upsilon\|_{\omega'}\cdot d\eta\wedge\omega' + 4\|\Upsilon\|^2_{\omega'}\cdot d\eta\wedge d\eta \\ &\quad + A(C-2)\frac{1}{\|\Upsilon\|_{\omega'}}\cdot d(\log\|\Upsilon\|_{\omega'})\wedge\mathrm{Re}\,(\Upsilon).\end{aligned} \tag{7.87}$$

A similar analysis to before using type decomposition gives the following:

**Theorem 7.2.11.** *Let $M$ be a contact Calabi–Yau 7-fold with contact form $\eta$, and nowhere-vanishing transverse holomorphic $(3,0)$-form $\Upsilon$. Let $\omega' \in [d\eta]_B$ be a transverse Kähler form. Suppose we have a family of contact forms $\widehat{\eta}$ and a family of compatible transverse $\mathrm{SU}(3)$-structures $(\widehat{\omega}', \widehat{\Upsilon})$ with $\widehat{\omega}' \in [d\widehat{\eta}]_B$ and initial conditions $\widehat{\eta}_0 = \eta$, $\widehat{\omega}'_0 = \omega'$, $\widehat{\Upsilon}_0 = \Upsilon$ satisfying the coupled differential equations*

$$\frac{\partial}{\partial t}\widehat{\eta} = 2\mathcal{L}_{\widehat{\nabla}_{\mathcal{D}}(\log \|\widehat{\Upsilon}\|_{\widehat{\omega}'})}\widehat{\eta} + \alpha, \tag{7.88}$$

$$\frac{\partial}{\partial t}\widehat{\omega}' = -2\mathcal{L}_{\widehat{\nabla}_{\mathcal{D}}(\log \|\widehat{\Upsilon}\|_{\widehat{\omega}'})}\widehat{\omega}' + \beta, \tag{7.89}$$

$$\frac{\partial}{\partial t}\widehat{\Upsilon} = 2\mathcal{L}_{\widehat{\nabla}_{\mathcal{D}}(\log \|\widehat{\Upsilon}\|_{\widehat{\omega}'})}\widehat{\Upsilon} + \gamma \tag{7.90}$$

*for some $\alpha \in \Omega^1(M)$, $\beta \in \Omega^2(M)$, and $\gamma \in \Omega^3(M)$. Let $\widehat{\varphi}$ be the family of $G_2$-structures on $M$ defined by*

$$\widehat{\varphi} = \mathrm{Re}\left(\frac{1}{\|\Upsilon\|_{\widehat{\omega}'}}\widehat{\Upsilon}\right) - \|\Upsilon\|_{\widehat{\omega}'} \cdot dr \wedge \widehat{\omega}', \tag{7.91}$$

*then $\widehat{\varphi}$ is a solution to the coflow (7.6) if and only if*

$$-2\alpha \wedge \mathrm{Im}\,(\widehat{\Upsilon}) = A(C-2)\frac{1}{\|\widehat{\Upsilon}\|_{\widehat{\omega}'}} \cdot d(\log \|\widehat{\Upsilon}\|_{\widehat{\omega}'}) \wedge \mathrm{Re}\,(\widehat{\Upsilon}), \tag{7.92}$$

$$\begin{aligned}
&- 2\widehat{\eta} \wedge \mathrm{Im}\,(\gamma) - \frac{1}{4}\beta \wedge \widehat{\omega}' \\
&= \Big[A(C-2) - 12C\|\widehat{\Upsilon}\|_{\widehat{\omega}'}\Big] \cdot \|\widehat{\Upsilon}\|_{\widehat{\omega}'} \cdot d(\log \|\widehat{\Upsilon}\|_{\widehat{\omega}'}) \wedge \widehat{\eta} \wedge \widehat{\omega}' \\
&\quad + 8\|\widehat{\Upsilon}\|^2_{\widehat{\omega}'} \cdot d(\log \|\widehat{\Upsilon}\|_{\widehat{\omega}'}) \wedge \widehat{\eta} \wedge d\widehat{\eta} \\
&\quad + \Big[A(C-2) - 6C\|\widehat{\Upsilon}\|_{\widehat{\omega}'}\Big] \cdot \|\widehat{\Upsilon}\|_{\widehat{\omega}'} \cdot d\widehat{\eta} \wedge \widehat{\omega}' + 4\|\widehat{\Upsilon}\|^2_{\widehat{\omega}'} \cdot d\widehat{\eta} \wedge d\widehat{\eta}.
\end{aligned} \tag{7.93}$$

# Chapter 8

# Convergence of Solutions

We revisit our solutions to the Laplacian flow and coflow on the trivial fibration $M = S^1 \times X$ from §6.3 and §7.2.1.

## 8.1 Parabolic Complex Monge–Ampère Flows

Recall that we had the following results:

**Theorem 6.3.1.** *Let $X$ be a Kähler Calabi–Yau threefold with Kähler form $\omega$ and nowhere-vanishing holomorphic $(3,0)$-form $\Upsilon$. Suppose we start the $G_2$-Laplacian coflow (7.1) on $M = S^1 \times X$ with initial data of the form*

$$\varphi_0 = \mathrm{Re}\,(\Upsilon) - dr \wedge \omega, \tag{6.33}$$

*then a solution to the flow exists for all time $t$ and is of the form*

$$\widehat{\varphi} = \mathrm{Re}\,(\widehat{\Upsilon}) - dr \wedge \widehat{\omega} \tag{6.34}$$

*where $\widehat{\omega} = \Theta^*\widetilde{\omega}$, $\widehat{\Upsilon} = \Theta^*\Upsilon$ with $\Theta$ being the 1-parameter family of diffeomorphisms generated by the (time-dependent) vector field*

$$Y = -2\widetilde{\nabla}(\|\Upsilon\|_{\widetilde{\omega}}^{-\frac{2}{3}}), \tag{6.35}$$

*and $\widetilde{\omega}$ solves the $MA^{\frac{1}{3}}$ flow on $X$ with initial condition $\omega_0 = \omega$.*

**Theorem 7.2.4.** *Let $X$ be a Kähler Calabi–Yau threefold with Kähler form $\omega$ and nowhere-vanishing holomorphic $(3,0)$-form $\Upsilon$. Suppose we start the $G_2$-Laplacian flow (6.1) on $M = S^1 \times X$ with initial data of the form*

$$\varphi_0 = \mathrm{Re}\left(\frac{1}{\|\Upsilon\|_\omega}\Upsilon\right) - \|\Upsilon\|_\omega \cdot dr \wedge \omega, \tag{7.44}$$

*then a solution to the flow exists for all time $t$ and is of the form*

$$\widehat{\varphi} = \mathrm{Re}\Big(\frac{1}{\|\widehat{\Upsilon}\|_{\widehat{\omega}}}\widehat{\Upsilon}\Big) - \|\widehat{\Upsilon}\|_{\widehat{\omega}} \cdot dr \wedge \widehat{\omega} \tag{7.45}$$

*where $\widehat{\omega} = \Theta^*\widetilde{\omega}$, $\widehat{\Upsilon} = \Theta^*\Upsilon$ with $\Theta$ being the 1-parameter family of diffeomorphisms generated by the (time-dependent) vector field*

$$Y = 2\widetilde{\nabla}(\log \|\Upsilon\|_{\widetilde{\omega}}), \tag{7.46}$$

*and $\widetilde{\omega}$ solves the Kähler–Ricci flow on $X$ with initial condition $\omega_0 = \omega$.*

These demonstrate the existence of solutions to the Laplacian flow and coflow respectively on $M = S^1 \times X$. It remains to discuss the convergence of these at infinity. This will rely on the theory of complex Monge–Ampère flows developed by Picard–Zhang [PZ20].

On a compact Kähler manifold $X$ of complex dimension $n$ and Kähler form $\omega$, we can consider the parabolic complex Monge–Ampère equation

$$\frac{\partial}{\partial t}u = H\Big(e^{-a} \cdot \frac{\det(\omega + \sqrt{-1}\partial\overline{\partial}u)}{\det \omega}\Big), \quad \omega + \sqrt{-1}\partial\overline{\partial}u > 0, \tag{8.1}$$

where $H : \mathbb{R}^+ \to \mathbb{R}$ is a smooth function with $H' > 0$ and $a \in C^\infty(X)$. We call a flow of the above form a (parabolic) complex Monge–Ampère flow.

Both the $\mathrm{MA}^{\frac{1}{3}}$ flow and the Kähler–Ricci flow can be realized in the above form. Recall that since we are working with a Kähler Calabi–Yau threefold $X$, we have the Kähler form $\omega$ and the nowhere-vanishing holomorphic $(3,0)$-form $\Upsilon$, which we shall use to define the function $a$. In particular, setting $H(\rho) = 12\rho^{\frac{1}{3}}$ and $a = 2\log\|\Upsilon\|_\omega$, we get the $\mathrm{MA}^{\frac{1}{3}}$ flow

$$\frac{\partial}{\partial t}u = 12\Big(e^{-2\log\|\Upsilon\|_\omega} \cdot \frac{\det(\omega + \sqrt{-1}\partial\overline{\partial}u)}{\det \omega}\Big)^{\frac{1}{3}} \tag{8.2}$$

from §6.3.

Likewise, if $H(\rho) = 4\log\rho$ and $a = 2\log\|\Upsilon\|_\omega$, we get

$$\frac{\partial}{\partial t}u = 4\log\Big(\frac{\det(\omega + \sqrt{-1}\partial\overline{\partial}u)}{\|\Upsilon\|_\omega^2 \cdot \det \omega}\Big), \tag{8.3}$$

which can be seen to be the (rescaled) Kähler–Ricci flow. Indeed, by setting $\widetilde{\omega} = \omega + \sqrt{-1}\partial\overline{\partial}u$, we can check that

$$\frac{\partial}{\partial t}u = 4\log\Big(\frac{\det(\omega + \sqrt{-1}\partial\overline{\partial}u)}{\|\Upsilon\|_\omega^2 \cdot \det \omega}\Big) = -4\log\|\Upsilon\|_{\widetilde{\omega}}, \tag{8.4}$$

and so

$$\frac{\partial}{\partial t}\widetilde{\omega} = \sqrt{-1}\partial\overline{\partial}\Big(\frac{\partial}{\partial t}u\Big) = -4\sqrt{-1}\partial\overline{\partial}(\log\|\Upsilon\|_{\widetilde{\omega}}) = -4\mathrm{Ric}(\widetilde{\omega}, J). \tag{8.5}$$

The result that we shall need is as follows:

**Theorem 8.1.1** (Picard–Zhang [PZ20])**.** *Let $X$ be a compact Kähler manifold of complex dimension $n$ with Kähler form $\omega$. Let $a \in C^\infty(X)$ and $H : \mathbb{R}^+ \to \mathbb{R}$ be a smooth function with $H' > 0$. Then the complex parabolic Monge–Ampère equation* (8.1) *has a smooth solution $u$ for all time $t$. Moreover, the function*

$$\widetilde{u} = u - \frac{1}{V}\cdot\int_X u\cdot\omega^n, \text{ where } V = \int_X \omega^n \tag{8.6}$$

*converges in $C^\infty(X, g)$ to a smooth function $\widetilde{u}_\infty$ satisfying*

$$(\omega + \sqrt{-1}\partial\overline{\partial}\widetilde{u}_\infty)^n = c_0 e^a \cdot \omega^n, \tag{8.7}$$

*where $c_0$ is a positive constant which can be determined by integration.*

*The evolving metrics $\widetilde{\omega} = \omega + \sqrt{-1}\partial\overline{\partial}u$ satisfy the uniform estimates*

$$C^{-1}\cdot g \le \widetilde{g} \le C\cdot g, \tag{8.8}$$

*and*

$$\|\nabla^k\widetilde{\omega}\|_g \le C_k \tag{8.9}$$

*for positive uniform constants $C$ and $C_k$ for $k \ge 0$.*

Applying this to our Calabi–Yau threefold $X$ and setting $a = 2\log\|\Upsilon\|_\omega$, we get a family of Kähler metrics $\widetilde{\omega} = \omega + \sqrt{-1}\partial\overline{\partial}u \in [\omega]$ which converge in each $C^k(X, g)$-norm to a limiting metric $\omega_{\mathrm{CY}} \in [\omega]$. This limiting metric satisfies

$$\omega^n_{\mathrm{CY}} = 2c_0\|\Upsilon\|_\omega \cdot \omega^n \tag{8.10}$$

and from this, we can deduce that

$$\|\Upsilon\|_{\omega_{\mathrm{CY}}} = 2c_0 \tag{8.11}$$

is constant. Hence $\omega_{\mathrm{CY}}$ is the unique Ricci-flat Kähler metric in the Kähler class $[\omega]$.

In addition to the uniform estimates from the theorem, we also require exponential convergence of the flow, which are given by the estimates

$$\left\|\frac{\partial}{\partial t}\nabla^k\widetilde{\omega}\right\|_g \leq C_k e^{-\lambda_k t} \tag{8.12}$$

for constants $C_k$ and $\lambda_k$. This was shown for the Kähler–Ricci flow by Phong–Sturm [PS06]. We shall prove this for general Monge–Ampère flows.

**Lemma 8.1.2.** *Let $X$ be a compact Kähler manifold of complex dimension $n$ with Kähler form $\omega$. Let $u$ be a solution to (8.1) for some $a \in C^\infty(X)$ and some smooth function $H : \mathbb{R}^+ \to \mathbb{R}$ with $H' > 0$.*

*Define the Kähler metrics $\widetilde{\omega} = \omega + \sqrt{-1}\partial\overline{\partial}u$ and $\widetilde{u}$ as in (8.6), then the flow converges exponentially.*

*Proof.* From [PZ20], we have the estimates

$$\|\widetilde{u}\|_{C^k(X,g)} \leq C_k, \quad \left|\frac{\partial}{\partial t}\widetilde{u}\right| \leq Ce^{-\lambda t}. \tag{8.13}$$

To get the decay of higher-order derivatives of $\widetilde{u}$, we use integration by parts. For example,

$$\begin{aligned}
\int_X \left\|\frac{\partial}{\partial t}\nabla\widetilde{u}\right\|_g^2 \cdot \omega^n &= \int_X \left|\frac{\partial}{\partial t}\widetilde{u}\right| \cdot \left|\frac{\partial}{\partial t}\Delta\widetilde{u}\right| \cdot \omega^n \\
&\leq Ce^{-\lambda t} \cdot \int_X \left|\frac{\partial}{\partial t}\Delta u\right| \cdot \omega^n \\
&\leq C \cdot \|\Delta H\|_{L^\infty(X,g)} \cdot e^{-\lambda t} \\
&\leq C_1 e^{-\lambda_1 t},
\end{aligned} \tag{8.14}$$

since $|\Delta H| \leq \|u\|_{C^4(X,g)} \leq C$. A similar calculation shows that

$$\int_X \left\|\frac{\partial}{\partial t}\nabla^k\widetilde{u}\right\|_g^2 \omega^n \leq C \cdot \|\nabla^{k+1}H\|_{L^\infty(X,g)} \cdot e^{-\lambda t} \leq C_k e^{-\lambda_k t}. \tag{8.15}$$

Using the Sobolev Embedding Theorem, we have

$$\left\|\frac{\partial}{\partial t}\nabla^k\widetilde{u}\right\|_{L^\infty(X,g)} \leq C_k e^{-\lambda_k t}. \tag{8.16}$$

Since $\widetilde{\omega} = \omega + \sqrt{-1}\partial\overline{\partial}\widetilde{u}$, we get (8.12). □

Using these results on the underlying flows on the base, we will be able to describe the long-term behaviour to the solutions described in Theorems 6.3.1 and 7.2.4.

## 8.2 Limits at Infinity

We return to $G_2$-geometry and discuss the convergence of the solutions $\widehat{\varphi}$. We tackle both the Laplacian flow and coflow cases simultaneously since their solutions were constructed in a similar manner.

Recall that in either case:

i) the flow is solved by the SU(3)-structure $(\widehat{\omega}, \widehat{\Upsilon}) = (\Theta^*\widetilde{\omega}, \Theta^*\Upsilon)$;

ii) the time-dependent family of Kähler triples $(\widetilde{\omega}, J, \widetilde{g})$ on $X$ come from a complex Monge–Ampère flow (either the $\mathrm{MA}^{\frac{1}{3}}$ flow or the Kähler–Ricci flow);

iii) the Kähler metrics $\widetilde{\omega}$ satisfy the estimates from the previous section and converge to the unique Ricci-flat Kähler metric $\omega_{\mathrm{CY}}$ in the Kähler class $[\omega]$ in each $C^k(X, g)$-norm;

iv) the diffeomorphisms $\Theta$ solve $\frac{\partial}{\partial t}\Theta = Y$, where either

$$Y = -2\widetilde{\nabla}(\|\Upsilon\|_{\widetilde{\omega}}^{-\frac{2}{3}}) \text{ or } Y = 2\widetilde{\nabla}(\log \|\Upsilon\|_{\widetilde{\omega}}). \tag{8.17}$$

To prove the convergence of $(\widehat{\omega}, \widehat{\Upsilon})$, we use a method similar to [LW19] to show that the maps $\Theta$ converge to a diffeomorphism $\Theta_\infty$.

Since the metrics $\widetilde{\omega}$ converge exponentially fast to $\omega_{\mathrm{CY}}$ along both the $\mathrm{MA}^{\frac{1}{3}}$ flow and the Kähler–Ricci flow, and since $\|\Upsilon\|_{\omega_{\mathrm{CY}}}$ is constant, it follows that $Y \to 0$ exponentially fast as well. Indeed, by (8.12) we see that

$$\|\nabla^k Y\|_g = \int_t^\infty \Big(\frac{\partial}{\partial s}\|\nabla^k Y\|_g\Big)\, ds \leq C \cdot \int_t^\infty e^{-\lambda s}\, ds. \tag{8.18}$$

Hence, for each $k \geq 0$, we have the estimates

$$\|\nabla^k Y\|_g \leq C_k e^{-\lambda_k t}. \tag{8.19}$$

We now consider the diffeomorphisms $\Theta$. For every point $p \in X$ and $t_1, t_2 \geq 0$, we have that

$$[t_1, t_2] \to X : t \mapsto \Theta_t(p) \tag{8.20}$$

defines a smooth path from $\Theta_{t_1}(p)$ to $\Theta_{t_2}(p)$. By our previous estimate (8.19) on $Y$, we have

$$d_g\big(\Theta_{t_1}(p), \Theta_{t_2}(p)\big) \leq \int_{t_1}^{t_2} \Big\|\frac{\partial}{\partial t}\Theta(p)\Big\|_g\, dt$$

$$\leq \int_{t_1}^{t_2} \|Y\|_g \, dt$$
$$\leq C \cdot \int_{t_1}^{t_2} e^{-\lambda t} \, dt. \tag{8.21}$$

It follows that the maps $\Theta$ converge uniformly with respect to the original metric $g$. Similarly, using the uniform estimates (8.8) and (8.9) we have that the maps $\Theta$ converge in each $C^k(X,g)$-norm. Thus, we have that they converge to some limit map $\Theta_\infty : X \to X$ as $t \to \infty$ in each $C^k(X,g)$-norm.

We now show that the pullback $\Theta_\infty^*$ is not degenerate. For this, we estimate

$$\begin{aligned}
\Big|\frac{\partial}{\partial t}\log\Big(\frac{\widehat{\Upsilon}\wedge\overline{\widehat{\Upsilon}}}{\Upsilon\wedge\overline{\Upsilon}}\Big)\Big| &= \Big|\frac{\partial}{\partial t}\log\Big(\frac{\Theta^*(\Upsilon\wedge\overline{\Upsilon})}{\Upsilon\wedge\overline{\Upsilon}}\Big)\Big| \\
&= \Big|\frac{1}{\Theta^*(\Upsilon\wedge\overline{\Upsilon})}\cdot\frac{\partial}{\partial t}(\Theta^*(\Upsilon\wedge\overline{\Upsilon}))\Big| \\
&= \Big|\Theta^*\Big(\frac{\mathcal{L}_Y(\Upsilon\wedge\overline{\Upsilon})}{\Upsilon\wedge\overline{\Upsilon}}\Big)\Big| \\
&\leq \sup_X\Big|\Big(\frac{\mathcal{L}_Y(\|\Upsilon\|_\omega^2\cdot\mathrm{vol})}{\|\Upsilon\|_\omega^2\cdot\mathrm{vol}}\Big)\Big| \\
&\leq \frac{|Y(\|\Upsilon\|_\omega^2)|}{\|\Upsilon\|_\omega} + \Big|\frac{d(Y \lrcorner \mathrm{vol})}{\mathrm{vol}}\Big| \leq Ce^{\lambda t}
\end{aligned} \tag{8.22}$$

using (8.19). As such

$$\begin{aligned}
\Big|\log\Big(\frac{\widehat{\Upsilon}\wedge\overline{\widehat{\Upsilon}}}{\Upsilon\wedge\overline{\Upsilon}}\Big)\Big| &\leq \int_0^t \Big|\frac{\partial}{\partial s}\log\Big(\frac{\widehat{\Upsilon}\wedge\overline{\widehat{\Upsilon}}}{\Upsilon\wedge\overline{\Upsilon}}\Big)\Big| \, ds \\
&\leq C\cdot\int_0^t e^{-\lambda s}\, ds \leq C
\end{aligned} \tag{8.23}$$

is bounded independently of $t$. It follows that

$$C^{-1}\cdot(\Upsilon\wedge\overline{\Upsilon}) \leq \Theta^*(\Upsilon\wedge\overline{\Upsilon}) \leq C\cdot(\Upsilon\wedge\overline{\Upsilon}) \tag{8.24}$$

and hence the pullback is uniformly non-degenerate. We see that $\det(\Theta_*)$ is bounded independently of $t$ and this estimate can be passed to the limit map $\Theta_\infty$.

By the Inverse Function Theorem, the limit map $\Theta_\infty$ is a local diffeomorphism. Since $\Theta_0 = \mathrm{Id}_X$ is the identity map and each $\Theta$ is a diffeomorphism which is isotopic to the identity, we have that $\Theta_\infty$ is a surjective local diffeomorphism homotopic to the identity. Further, $X$ is compact, and so $\Theta_\infty$ is a covering map. Lastly, since $\Theta_\infty$ is homotopic to the identity, it has degree 1 and is thus injective. From this, we conclude that $\Theta_\infty$ is indeed a diffeomorphism.

Finally, we have that $\Theta \to \Theta_\infty$ and $\widetilde{\omega} \to \omega_{\mathrm{CY}}$ with respect to the background metric $g$. It follows that $\widehat{\omega} = \Theta^*\widetilde{\omega} \to \Theta_\infty^*\omega_{\mathrm{CY}}$ and $\widehat{\Upsilon} = \Theta^*\Upsilon \to \Theta_\infty^*\Upsilon$ also with respect to $g$. As such, our solution on $M$ converges to

$$\widehat{\varphi} \to \widehat{\varphi}_\infty = \mathrm{Re}\,(\Theta_\infty^*\Upsilon) - dr \wedge (\Theta_\infty^*\omega_{\mathrm{CY}}) \tag{8.25}$$

in the Laplacian flow case and

$$\widehat{\varphi} \to \widehat{\varphi}_\infty = \mathrm{Re}\left(\Theta_\infty^*\left[\frac{1}{\|\Upsilon\|_{\omega_{\mathrm{CY}}}}\Upsilon\right]\right) - dr \wedge \left(\Theta_\infty^*(\|\Upsilon\|_{\omega_{\mathrm{CY}}} \cdot \omega_{\mathrm{CY}})\right) \tag{8.26}$$

in the Laplacian coflow case.

Since $\|\Upsilon\|_{\omega_{\mathrm{CY}}}$ is constant, we see that in either case, $\widehat{\varphi}_\infty$ defines a torsion-free $G_2$-structure.

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# Appendices

# Appendix A

# Identities for Hermitian Metrics and Chern Connections

We list some useful identities that will be used often. As before, we let $X$ be a Calabi–Yau $n$-fold with nowhere-vanishing holomorphic $(n,0)$-form $\Upsilon$. Much of this appendix is based on §2.2 of [PPZ18b].

Given a Hermitian metric $\omega = \sqrt{-1} g_{j\bar{k}} dz^j \wedge d\bar{z}^k$, we define its torsion tensors $T$ and $\overline{T}$ by

$$T = \sqrt{-1}\partial\omega \text{ and } \overline{T} = -\sqrt{-1}\bar{\partial}\omega. \tag{A.1}$$

This defines a $(2,1)$- and $(1,2)$-form respectively.

We can write these in local coordinates by

$$T = \frac{1}{2} T_{mj\bar{k}} dz^m \wedge dz^j \wedge d\bar{z}^{\bar{k}} \text{ and } \overline{T} = \frac{1}{2} \overline{T}_{j\overline{m}\bar{k}} dz^j \wedge d\bar{z}^{\overline{m}} \wedge d\bar{z}^{\bar{k}} \tag{A.2}$$

where

$$T_{mj\bar{k}} = \partial_j g_{m\bar{k}} - \partial_m g_{j\bar{k}} \text{ and } \overline{T}_{j\overline{m}\bar{k}} = \overline{\partial}_{\bar{k}} g_{j\overline{m}} - \overline{\partial}_{\overline{m}} g_{j\bar{k}}. \tag{A.3}$$

It is convenient to define the tensors

$$T_m = g^{j\bar{k}} T_{mj\bar{k}} \text{ and } \overline{T}_{\overline{m}} = g^{\bar{j}k} \overline{T}_{j\overline{m}\bar{k}}. \tag{A.4}$$

We have the following result of Li–Yau [LY05], stated in a more general form.

**Lemma A.0.1.** *Let $X$ be a Calabi–Yau $n$-fold with nowhere-vanishing holomorphic $(n,0)$-form $\Upsilon$. Suppose $\omega$ is a Hermitian metric on $X$. Then the following are equivalent:*

*i) $\omega$ satisfies the conformally balanced condition $d\big(\|\Upsilon\|_\omega^a \omega^{n-1}\big) = 0$ for some constant $a$;*

*ii)* $d^\dagger\omega = \sqrt{-1}(\overline{\partial} - \partial)\log\|\Upsilon\|^a_\omega$*; and*

*iii)* $T_m = \partial_m \log\|\Upsilon\|^a_\omega$, $\overline{T_{\overline{m}}} = \overline{\partial}_{\overline{m}}\log\|\Upsilon\|^a_\omega$.

*Proof.* We can expand the $(n, n-1)$-part of the conformally balanced condition to get

$$\partial\log\|\Upsilon\|^a_\omega \wedge \omega^{n-1} + (n-1)\partial\omega \wedge \omega^{n-2} = 0. \tag{A.5}$$

One can also check that for a $(2,1)$-form $A = \frac{1}{2}A_{mj\overline{k}}dz^m \wedge dz^j \wedge d\overline{z}^k$, we have

$$A \wedge \omega^{n-2} = -\sqrt{-1}(g^{j\overline{k}}A_{mj\overline{k}}dz^m) \wedge \frac{\omega^{n-1}}{n-1}. \tag{A.6}$$

In particular, taking $A$ to be the torsion tensor $T = \sqrt{-1}\partial\omega$, we see that the conformally balanced condition is equivalent to

$$\begin{aligned}\partial\log\|\Upsilon\|^a_\omega \wedge \omega^{n-1} &= -(n-1)\partial\omega \wedge \omega^{n-2} \\ &= \sqrt{-1}(n-1)T \wedge \omega^{n-2} \\ &= T_m dz^m \wedge \omega^{n-1}.\end{aligned} \tag{A.7}$$

That is

$$(\partial\log\|\Upsilon\|^a_\omega - T_m dz^m) \wedge \omega^{n-1} = 0, \tag{A.8}$$

and since the wedge product by $\omega^{n-1}$ is an isomorphism we get

$$\partial\log\|\Upsilon\|^a_\omega - T_m dz^m. \tag{A.9}$$

By taking complex conjugates, we see that $i)$ and $iii)$ are equivalent.

To get that the equivalence between $ii)$ and $iii)$, we use the expressions of the adjoints of $\partial$ and $\overline{\partial}$ on a $(1,1)$-form $A$:

$$(\overline{\partial}^\dagger A)_m = g^{j\overline{k}}(\nabla_j A_{m\overline{k}} - T_j A_{m\overline{k}}) \text{ and } (\partial^\dagger A)_{\overline{m}} = -g^{j\overline{k}}(\overline{\nabla}_{\overline{k}} A_{j\overline{m}} - \overline{T}_{\overline{k}} A_{j\overline{m}}). \tag{A.10}$$

If we set $A = \omega$, then $A_{j\overline{k}} = \sqrt{-1}g_{j\overline{k}}$ which gives

$$(\overline{\partial}^\dagger\omega)_m = -\sqrt{-1}g^{j\overline{k}}(\nabla_j g_{m\overline{k}} - T_j g_{m\overline{k}}) = -\sqrt{-1}T_m. \tag{A.11}$$

Likewise

$$(\partial^\dagger\omega) = \sqrt{-1}\overline{T_{\overline{m}}}. \tag{A.12}$$

Putting this together, we have that given *iii*),

$$\begin{aligned} d^\dagger\omega &= \partial^\dagger\omega + \overline{\partial}^\dagger\omega \\ &= -\sqrt{-1}T + \sqrt{-1}\overline{T} \\ &= \sqrt{-1}(\overline{\partial} - \partial)\log\|\Upsilon\|^a_\omega. \end{aligned} \tag{A.13}$$

Conversely, if $d^\dagger\omega = \sqrt{-1}(\overline{\partial} - \partial)\log\|\Upsilon\|^a_\omega$, then by type decomposition we have

$$-\sqrt{-1}T = \partial^\dagger\omega = -\sqrt{-1}\partial\log\|\Upsilon\|^a_\omega, \tag{A.14}$$

as desired. □

We are interested in the case where the constant $a = 1$. Taking covariant derivatives, we can check that

$$\nabla\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) = -\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot T \text{ and } \overline{\nabla}\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) = -\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot\overline{T}. \tag{A.15}$$

In particular, repeated application of this yields

$$\begin{aligned} &\nabla^m\overline{\nabla}^l\Big(\frac{1}{2\|\Upsilon\|_\omega}\Big) \\ &= \Big(\frac{1}{2\|\Upsilon\|_\omega}\Big)\cdot \sum_{\substack{i_1+\ldots+i_r+(r-s)=m \\ j_1+\ldots+j_s=l}} (\nabla^{i_1}\overline{\nabla}^{j_1}\overline{T}) * \ldots * (\nabla^{i_s}\overline{\nabla}^{j_s}\overline{T}) \\ &\qquad\qquad * (\nabla^{i_{s+1}}T) * \ldots * (\nabla^{i_r}T). \end{aligned} \tag{A.16}$$

We end with some general commutator identities covariant derivatives. If $A$ is a generic tensor, then

$$\begin{aligned} &\nabla^m\overline{\nabla}^l(\Delta_R A) \\ &= \Delta_R(\nabla^m\overline{\nabla}^l A) + \sum_{i=0}^m\sum_{j=0}^l(\nabla^{m-i}\overline{\nabla}^{l-j}A) * (\nabla^i\overline{\nabla}^j\mathrm{Rm}) \\ &\quad + \sum_{i=0}^m\sum_{j=0}^l(\nabla^{m-i}\overline{\nabla}^{l+1-j}A) * (\nabla^i\overline{\nabla}^j T) \\ &\quad + \sum_{i=0}^m\sum_{j=0}^l(\nabla^{m+1-i}\overline{\nabla}^{l-j}A) * (\nabla^i\overline{\nabla}^j\overline{T}), \end{aligned} \tag{A.17}$$

also

$$\overline{\nabla}^l \nabla^m A$$
$$= \sum_{r=0}^{\min(m,l)} \sum_{\substack{i_0+\ldots+i_r=m-r \\ j_0+\ldots+j_r=l-r}} (\nabla^{i_0}\overline{\nabla}^{j_0} A) * (\nabla^{i_1}\overline{\nabla}^{j_1}\mathrm{Rm}) * \ldots * (\nabla^{i_r}\overline{\nabla}^{j_r}\mathrm{Rm}). \tag{A.18}$$

Finally, we have the Divergence Theorem for the Chern connection:

$$\int_X \nabla_i V^i = \int_X T_i \cdot V^i \text{ and } \int_X \overline{\nabla}_{\bar{j}} V^{\bar{j}} = \int_X \overline{T}_{\bar{j}} \cdot V^{\bar{j}}. \tag{A.19}$$

# Appendix B

# The (Local) Maps $\Phi_t$

We collect several results regarding the maps $\Phi_t : \mathbb{C}^4 \setminus \{0\} \to \mathbb{C}^4$ introduced in §1.2.1. First, recall that $\Phi_t$ was defined by

$$\Phi_t(z) = z + \frac{t\overline{z}}{2\|z\|^2} \tag{B.1}$$

Furthermore, for $R \neq 0$, we had scaling maps $S_R : \mathbb{C}^4 \to \mathbb{C}^4$ given by

$$S_R(z) = R^{\frac{3}{2}} \cdot z. \tag{B.2}$$

In particular, we have the relation

$$\Phi_t = S_{t^{\frac{1}{3}}} \circ \Phi_1 \circ S_{t^{-\frac{1}{3}}}. \tag{B.3}$$

Since the scaling maps are smooth, we can restrict our focus to the map $\Phi_1$. In particular, we will show that the restriction

$$\Phi_1 : \left\{ z \in V_0 \mid \|z\|^2 > \frac{1}{2} \right\} \to \{ z \in V_1 \mid \|z\|^2 > 1 \} \tag{B.4}$$

is a diffeomorphism, where the spaces $V_t$ are defined by

$$V_t = \left\{ z \in \mathbb{C}^4 \mid \sum_{j=1}^{4} z_j^2 = t \right\}. \tag{B.5}$$

Let $z \in V_0$ and so $z \cdot z = \overline{z} \cdot \overline{z} = 0$. Taking norms, we have that

$$\begin{aligned} \|\Phi_1(z)\|^2 &= \left( z + \frac{\overline{z}}{2\|z\|^2} \right) \cdot \left( \overline{z} + \frac{z}{2\|z\|^2} \right) \\ &= \|z\|^2 + \frac{z \cdot z}{2\|z\|^2} + \frac{\overline{z} \cdot \overline{z}}{2\|z\|^2} + \frac{\|z\|^2}{4\|z\|^4} \\ &= \|z\|^2 \cdot \left( 1 + \frac{1}{4\|z\|^4} \right). \end{aligned} \tag{B.6}$$

We can check where each level set of the cone $V_0$ is mapped to by $\Phi_1$ by considering the function

$$f(x) = x \cdot \Big(1 + \frac{1}{4x^2}\Big), \qquad x > 0. \tag{B.7}$$

It can be seen that $f$ is strictly increasing on $(\frac{1}{2}, \infty)$. From this, we can see that $\Phi_1$ is injective on $\{z \in V_0 \mid \|z\|^2 > \frac{1}{2}\}$. Indeed, suppose that $\Phi_1(z) = \Phi_1(z')$ where $\|z\|^2 > \frac{1}{2}$ and $\|z'\|^2 > \frac{1}{2}$. The restriction of the domain then implies that $\|z\|^2 = \|z'\|^2$. Splitting the equation $\Phi_1(z) = \Phi_1(z')$ into real and imaginary components and a straightforward computation shows that $z = z'$.

Next, we find an inverse of the restriction $\Phi_1$. We first note that

$$g(x) = \frac{1}{2}(x + \sqrt{x^2 - 1}), \qquad x > 1, \tag{B.8}$$

defines an inverse for $f : (\frac{1}{2}, \infty) \to (1, \infty)$.

Let $w \in V_1$ with $\|w\|^2 = B > 1$. By direct calculation, one can check that if

$$z_j = \Big(\frac{2g(B)}{2g(B)+1}\Big) \cdot \mathrm{Re}\,(w_j) - \sqrt{-1}\Big(\frac{2g(B)}{2g(B)-1}\Big) \cdot \mathrm{Im}\,(w_j), \tag{B.9}$$

then

$$z \in V_0, \qquad \|z\|^2 = g(B) > \frac{1}{2}, \text{ and } \Phi_1(z) = w. \tag{B.10}$$

It follows that $\Phi_1$ is a bijection from $\{z \in V_0 \mid \|z\|^2 > \frac{1}{2}\}$ to $\{z \in V_1 \mid \|z\|^2 > 1\}$. Since the coordinate expressions for $\Phi_1$ and its inverse are smooth on their domain, we see that $\Phi_1$ is a diffeomorphism. By composing with the scaling maps and writing in terms of the radius function $r(z) = \|z\|^{\frac{2}{3}}$, we have

**Proposition B.0.1.** *The map*

$$\Phi_t : \Big\{z \in V_0 \mid r(z) > \Big(\frac{|t|}{2}\Big)^{\frac{1}{3}}\Big\} \to \Big\{z \in V_t \mid r(z) > |t|^{\frac{1}{3}}\Big\} \tag{B.11}$$

*is a diffeomorphism.*

# Appendix C

# Sasakian Manifolds

We review some basics on Sasakian manifolds and discuss some useful results involving deformations of Sasakian structures. For more details, we refer the reader to *e.g.,* [Bla76, BG08, BGS08].

## C.1 Contact and Almost-Contact Structures

We begin with a discussion on contact and almost-contact structures on odd-dimensional manifolds.

**Definition C.1.1.** An contact structure on a $(2n+1)$-dimensional manifold $M$ is a 1-form $\eta$, called the contact form, such that

$$\eta \wedge (d\eta)^n \neq 0. \tag{C.1}$$

On a contact manifold $(M, \eta)$, it can be shown that there exists a unique vector field $\xi$ called the Reeb vector field such that

$$\eta(\xi) = 1 \text{ and } \xi \lrcorner (d\eta) = 0. \tag{C.2}$$

Using the Reeb vector field $\xi$, we obtain a 1-dimensional foliation $\mathcal{F}_\xi$, and its dual 1-form $\eta$ determines a codimension 1 subbundle $\mathcal{D} = \ker \eta$ of $TM$. We have a canonical splitting

$$TM = \mathcal{D} \oplus L\xi, \tag{C.3}$$

where $L\xi$ is the line bundle spanned by $\xi$.

Given a contact structure $\eta$ on $M$, we would like to define a Riemannian metric $g$ in a suitable manner.

**Definition C.1.2.** An almost-contact structure on a $(2n+1)$-dimensional manifold $M$ consists of a vector field $\xi$, a 1-form $\eta$, and an endomorphism $\Phi$ of $TM$ such that

$$\eta(\xi) = 1 \text{ and } \quad \Phi^2 = -\mathrm{Id}_{TM} + \xi \otimes \eta. \tag{C.4}$$

As a consequence of these relations, we also have the following which can be found in [Bla76]:

**Lemma C.1.3.** *If $(\xi, \eta, \Phi)$ is an almost-contact structure on $M$, then*

$$\Phi(\xi) = 0 \text{ and } \eta \circ \Phi = 0. \tag{C.5}$$

*Additionally, the endomorphism $\Phi$ has pointwise rank $2n$.*

*Proof.* The conditions in (C.4) imply that

$$\Phi^2(\xi) = -\xi + \eta(\xi) \cdot \xi = 0. \tag{C.6}$$

Thus, either $\Phi(\xi) = 0$ or $\Phi(\xi)$ is a non-trivial eigenvector of $\Phi$ with eigenvalue 0. The conditions again imply that

$$0 = \Phi^2\big(\Phi(\xi)\big) = -\Phi(\xi) + \eta\big(\Phi(\xi)\big) \cdot \xi, \tag{C.7}$$

that is,

$$\Phi(\xi) = \eta\big(\Phi(\xi)\big) \cdot \xi. \tag{C.8}$$

If $\Phi(\xi)$ is a non-trivial eigenvector of $\Phi$, then we must have $\eta\big(\Phi(\xi)\big) \neq 0$, and hence

$$0 = \Phi^2(\xi) = \eta\big(\Phi(\xi)\big) \cdot \Phi(\xi) = \big[\eta\big(\Phi(\xi)\big)\big]^2 \cdot \xi \neq 0, \tag{C.9}$$

which is a contradiction. As such, $\Phi(\xi) = 0$.

For the other relation, the we see that by the second condition in (C.4)

$$\begin{aligned} \eta\big(\Phi(Y)\big) \cdot \xi &= \Phi^3(Y) + \Phi(Y) \\ &= -\Phi(Y) + \Phi\big(\eta(Y) \cdot \xi\big) + \Phi(Y) = 0 \end{aligned} \tag{C.10}$$

for any vector field $Y$. It follows that $\eta \circ \Phi = 0$.

To check the rank of $\Phi$, we note that since $\Phi(\xi) = 0$, we must have $\operatorname{rank} \Phi < 2n+1$. Suppose $\Phi(Y) = 0$. We then have

$$0 = \Phi^2(Y) = -Y + \eta(Y) \cdot \xi \tag{C.11}$$

and hence $Y$ is some multiple of $\xi$. Thus, $\operatorname{rank} \Phi = 2n$. □

A Riemannian metric $g$ on $M$ is compatible with the almost-contact structure if

$$g\big(\Phi(Y), \Phi(Z)\big) = g(Y, Z) - \eta(Y) \cdot \eta(Z), \tag{C.12}$$

for any vector fields $Y, Z$ on $M$. In this case, the quadruple $(\xi, \eta, \Phi, g)$ is called a almost-contact metric structure. By setting $Z = \xi$, we can also see that

$$\eta(Y) = g(Y, \xi). \tag{C.13}$$

Given an almost-contact metric structure, we can define a fundamental 2-form $\omega$ by

$$\omega(Y, Z) = g\big(\Phi(Y), Z\big). \tag{C.14}$$

Indeed, this is skew since

$$\begin{aligned} g\big(\Phi(Y), Z\big) &= g\big(\Phi^2(Y), \Phi(Z)\big) - \eta\big(\Phi(Y)\big) \cdot \eta(Z) \\ &= -g\big(Y, \Phi(Z)\big) + \eta(Y) \cdot g\big(\xi, \Phi(Z)\big) \\ &= -g\big(\Phi(Z), Y\big) + \eta(Y) \cdot \Big[g\big(\Phi(\xi), \Phi^2(Z)\big) + \eta(\xi) \cdot \eta\big(\Phi(Z)\big)\Big] \\ &= -g\big(\Phi(Z), Y\big). \end{aligned} \tag{C.15}$$

Since $\operatorname{rank} \Phi = 2n$, one can additionally check that

$$\eta \wedge \omega^n \neq 0. \tag{C.16}$$

**Proposition C.1.4.** *Let $M$ be a $(2n+1)$-dimensional contact manifold with a contact form $\eta$, then there exists an almost-contact metric structure $(\xi, \eta, \Phi, g)$ such that the fundamental 2-form $\omega$ satisfies $\omega = d\eta$.*

A quadruple $(\xi, \eta, \Phi, g)$ compatible with a contact structure $\eta$ is called a contact metric structure. Additionally, contact metric structures are not unique.

Consider the cone $C(M) = \mathbb{R}^+ \times M$ endowed with the metric

$$g_C = dr \otimes dr + r^2 \cdot g. \tag{C.17}$$

We can define an almost-complex structure $J_C$ on $C(M)$ by

$$J_C(Y) = \Phi(Y) + \eta(Y) \cdot r\frac{\partial}{\partial r}, \quad J_C\Big(r\frac{\partial}{\partial r}\Big) = -\xi. \tag{C.18}$$

An almost-contact metric structure $(\xi, \eta, \Phi, g)$ is called normal if $\big(C(M), J_C\big)$ is a complex manifold. In this case, the almost-complex structure $J = \Phi|_{\mathcal{D}}$ is integrable.

We can now define a Sasakian structure:

**Definition C.1.5.** A Sasakian structure $\mathcal{S}$ on a $(2n+1)$-dimensional contact manifold $M$ with contact form $\eta$ is a normal contact metric structure $(\xi, \eta, \Phi, g)$.

Given a Sasakian structure $\mathcal{S}$, one can check that the restriction of the metric $g$ to $\mathcal{D}$ is a positive-definite symmetric form which we refer to as the transverse Kähler metric. The transverse Kähler form of this metric is $\omega = d\eta$ and the Sasakian metric $g$ can be written as

$$g = d\eta \circ (\mathrm{Id} \otimes \Phi) + \eta \otimes \eta. \tag{C.19}$$

## C.2 Basic Forms and Deformations of Sasakian Structures

**Definition C.2.1.** A $k$-form $\beta$ on a contact manifold $M$ is called basic if

$$\xi \lrcorner \beta = 0, \quad \mathcal{L}_\xi \beta = 0, \tag{C.20}$$

where $\xi$ is the Reeb vector field. The set of basic $k$ forms will be denoted by $\Omega^k_B(M)$.

Using Cartan's Magic Formula, one can see that the Lie derivative condition is equivalent to $\xi \lrcorner (d\beta) = 0$, and so the exterior derivative preserves basic forms. Basic cohomology classes, denoted by $[\cdot]_B$, can be defined in the usual way with the appropriate restrictions. Our previous discussion shows that the transverse Kähler form $\omega$ is basic, and so its curvature tensors must also be basic. This in part gives us a notion of basic Chern classes $c^B_k(M)$.

One can check that given a Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ on $M$, then

$$J\big(\Omega^k_B(M)\big) = \Omega^k_B(M). \tag{C.21}$$

As such, we can decompose the complexified space in the expected manner:

$$\Omega^k_B(M) \otimes \mathbb{C} = \bigoplus_{p+q=k} \Omega^{p,q}_B(M). \tag{C.22}$$

From this, we have the usual $\partial$ and $\overline{\partial}$ operators, giving rise to a transverse Hodge theory which largely mirrors the normal setting.

Given a Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ on $M$, we wish to deform it in a manner that preserves the Reeb vector field $\xi$ and the underlying transverse structure. We let

$$\mathfrak{F}(\xi) = \{\text{Sasakian structures } \mathcal{S}' = (\xi', \eta', \Phi', g') \mid \xi' = \xi\}. \tag{C.23}$$

Given two Sasakian structures $\mathcal{S}, \mathcal{S}' \in \mathfrak{F}(\xi)$, with contact forms $\eta$ and $\eta'$ respectively, we have that their difference $\zeta = \eta - \eta'$ is basic. As such, $[d\eta']_B = [d\eta]_B$ and hence all Sasakian structures in $\mathfrak{F}(\xi)$ have the same basic cohomology class.

Consider the quotient bundle $\nu(\mathcal{F}_\xi) = TM/L\xi$. This bundle has an induced complex structure $\overline{J}$ and quotient map $\pi_\nu : TM \to \nu(\mathcal{F}_\xi)$. We define the subset $\mathfrak{F}(\xi, \overline{J}) \subseteq \mathfrak{F}(\xi)$ to be the subsect of Sasakians structures $(\xi', \eta', \Phi', g') \in \mathfrak{F}(\xi)$ such that the diagram

$$\begin{array}{ccc} TM & \xrightarrow{\Phi'} & TM \\ \downarrow{\scriptstyle \pi_\nu} & & \downarrow{\scriptstyle \pi_\nu} \\ \nu(\mathcal{F}_\xi) & \xrightarrow{\overline{J}} & \nu(\mathcal{F}_\xi) \end{array} \tag{C.24}$$

commutes. These are the Sasakian structures with the same transverse holomorphic structure $\overline{J}$.

The following result is a transverse version of the $\sqrt{-1}\partial\overline{\partial}$-Lemma.

**Proposition C.2.2** (El Kacimi-Alaoui [EKA90]). *Let $M$ be a compact Sasakian manifold and let $\omega$ and $\omega'$ be real basic closed $(1,1)$-forms such that $[\omega]_B = [\omega']_B$. Then there exists a basic function $\zeta$ such that*

$$\omega' = \omega + \sqrt{-1}\partial\overline{\partial}\zeta = \omega + dd^c\zeta, \tag{C.25}$$

*where $d^c = \frac{\sqrt{-1}}{2}(\overline{\partial} - \partial)$.*

As in the Kähler case, the basic 2-form $\omega = d\eta$ can be written locally in terms of a basic potential function $\zeta$, *i.e.,* $d\eta = dd^c\zeta$, and so Sasakian geometry is locally determined by a basic potential.

There exists a characterization of the space of Sasakian structures with fixed Reeb vector field $\xi$ and transverse holomorphic structure $\overline{J}$ as an affine space. We will not require the full description, but will use the following:

**Definition C.2.3.** Given a Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g) \in \mathfrak{F}(\xi, \overline{J})$ on $M$, a transformation of the form $\eta \mapsto \eta' = \eta + d^c\zeta$ for a basic function $\zeta$ is an instance of a deformation of type II. Such a transformation induces an endomorphism $\Phi'$ and metric $g'$ by

$$\Phi' = \Phi - \big(\xi \otimes (d^c\zeta)\big) \circ \Phi, \tag{C.26}$$

$$g' = d\eta' \circ \big(\mathrm{Id} \otimes \Phi\big) + \eta' \otimes \eta'. \tag{C.27}$$

The ensuing Sasakian structure $\mathcal{S}' = (\xi, \eta', \Phi', g')$ also lies in $\mathfrak{F}(\xi, \overline{J})$.

# Appendix D

# Structures Induced from $G_2$-Structures

In this appendix, we overview the computations for the associated structures given a $G_2$-structure $\varphi$. In particular, we work with the Ansätze introduced in §5.3.1 and §5.3.2.

## D.1 Hermitian and Riemannian Metrics

Before we begin, we first note an important relation between a Hermitian metric and its associated Riemannian metric.

Let $g$ be a Riemannian metric and $J$ be an almost-complex structure on a $2n$-dimensional manifold $M$ such that $g$ is Hermitian ( *i.e.,* $g(Y,Z) = g(JY,JZ)$). We have local real coordinates $x^1, y^1, \ldots, x^n, y^n$ such that

$$J\Big(\frac{\partial}{\partial x^j}\Big) = \frac{\partial}{\partial y^j}, \quad J\Big(\frac{\partial}{\partial y^j}\Big) = -\frac{\partial}{\partial x^j}. \tag{D.1}$$

In these coordinates, we have

$$g\Big(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\Big) = g\Big(J\Big(\frac{\partial}{\partial x^j}\Big), J\Big(\frac{\partial}{\partial x^k}\Big)\Big) = g\Big(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}\Big) \tag{D.2}$$

and

$$\begin{aligned} g\Big(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^k}\Big) &= g\Big(J\Big(\frac{\partial}{\partial x^j}\Big), J\Big(\frac{\partial}{\partial y^k}\Big)\Big) \\ &= g\Big(\frac{\partial}{\partial y^j}, -\frac{\partial}{\partial x^k}\Big) = -g\Big(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial x^k}\Big). \end{aligned} \tag{D.3}$$

Expanding everything out, we can write

$$g = g\Big(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\Big) dx^j \otimes dx^k + g\Big(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^k}\Big) dx^j \otimes dy^k$$

$$+ g\Big(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial x^k}\Big) dy^j \otimes dx^k + g\Big(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}\Big) dy^j \otimes dy^k. \tag{D.4}$$

In matrix form, (reordering the basis to $x^1, \ldots, x^n, y^1, \ldots, y^n$) we have

$$g_{jk} = \begin{bmatrix} C & D \\ -D & C \end{bmatrix} \tag{D.5}$$

for some real $n \times n$ matrices $C, D$ with $C$ symmetric and $D$ skew-symmetric.

Complexifying, we have complex coordinates $z^j = x^j + \sqrt{-1}y^j$, $\overline{z}^{\overline{j}} = x^j - \sqrt{-1}y^j$ with

$$dz^j = dx^j + \sqrt{-1}dy^j, \quad d\overline{z}^{\overline{j}} = dx^j - \sqrt{-1}dy^j, \tag{D.6}$$

$$\frac{\partial}{\partial z^j} = \frac{1}{2}\Big(\frac{\partial}{\partial x^j} - \sqrt{-1}\frac{\partial}{\partial y^j}\Big), \quad \frac{\partial}{\partial \overline{z}^{\overline{j}}} = \frac{1}{2}\Big(\frac{\partial}{\partial x^j} + \sqrt{-1}\frac{\partial}{\partial y^j}\Big). \tag{D.7}$$

One can then check that

$$\begin{aligned} g\Big(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \overline{z}^{\overline{k}}}\Big) &= g\Big(\frac{1}{2}\Big(\frac{\partial}{\partial x^j} - \sqrt{-1}\frac{\partial}{\partial y^j}\Big), \frac{1}{2}\Big(\frac{\partial}{\partial x^k} + \sqrt{-1}\frac{\partial}{\partial y^k}\Big)\Big) \\ &= \frac{1}{4}\Big[g\Big(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\Big) + g\Big(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}\Big) \\ &\qquad + \sqrt{-1}g\Big(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^k}\Big) - \sqrt{-1}g\Big(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial x^k}\Big)\Big] \\ &= \frac{1}{2}\Big[g\Big(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\Big) + \sqrt{-1}g\Big(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^k}\Big)\Big], \end{aligned} \tag{D.8}$$

which in matrix form means

$$g_{j\overline{k}} = \frac{1}{2}\left[C + \sqrt{-1}D\right]. \tag{D.9}$$

Since

$$\begin{aligned} &\begin{bmatrix} I_n & 0 \\ -\sqrt{-1}I_n & I_n \end{bmatrix} \begin{bmatrix} C & D \\ -D & C \end{bmatrix} \begin{bmatrix} I_n & 0 \\ \sqrt{-1}I_n & I_n \end{bmatrix} \\ &= \begin{bmatrix} C & D \\ -D - \sqrt{-1}C & C - \sqrt{-1}D \end{bmatrix} \begin{bmatrix} I_n & 0 \\ \sqrt{-1}I_n & I_n \end{bmatrix} \\ &= \begin{bmatrix} C + \sqrt{-1}D & D \\ 0 & C - \sqrt{-1}D \end{bmatrix}, \end{aligned}$$

we have

$$\det \begin{bmatrix} C & D \\ -D & C \end{bmatrix} = \det(C + \sqrt{-1}D)\det(C - \sqrt{-1}D). \tag{D.10}$$

By the symmetry of $C$ and the skew-symmetry of $D$, we see that $C+\sqrt{-1}D$ is Hermitian and so has real determinant. It follows that

$$\sqrt{\det g_{jk}} = 2^n \det g_{j\overline{k}}. \tag{D.11}$$

Defining $\omega(Y,Z) = g(JY,Z)$, we have

$$\omega = \sqrt{-1} g_{j\overline{k}} dz^j \wedge d\overline{z}^{\overline{k}}. \tag{D.12}$$

Thus

$$\begin{aligned} \frac{\omega^n}{n!} &= (\sqrt{-1})^n \det g_{j\overline{k}} \cdot dz^1 \wedge d\overline{z}^1 \wedge \ldots \wedge dz^n \wedge d\overline{z}^n \\ &= 2^n \det g_{j\overline{k}} \cdot dx^1 \wedge dy^1 \wedge \ldots \wedge dx^n \wedge dy^n \\ &= \sqrt{\det g_{jk}} \cdot dx^1 \wedge dy^1 \wedge \ldots dx^n \wedge dy^n \\ &= \mathrm{vol}. \end{aligned}$$

## D.2 Local Computations

We first note the following from *e.g.,* [Kar09, Kar20]: Let $\varphi$ be a $G_2$-structure on a 7-fold $M$. Given local coordinates, $x^1, \ldots, x^7$, we define a tensor $B$ by

$$B_{jk} dx^1 \wedge \ldots \wedge dx^7 = \left(\frac{\partial}{\partial x^j} \lrcorner \varphi\right) \wedge \left(\frac{\partial}{\partial x^k} \lrcorner \varphi\right) \wedge \varphi. \tag{D.13}$$

From the relation (5.2), we see that this is also

$$B_{jk} dx^1 \wedge \ldots \wedge dx^7 = -6(g_\varphi)_{jk} \mathrm{vol}_\varphi = -6(g_\varphi)_{jk} \sqrt{\det(g_\varphi)} \cdot dx^1 \wedge \ldots \wedge dx^7. \tag{D.14}$$

The metric $(g_\varphi)_{jk}$ and volume form $\mathrm{vol}_\varphi$ can be extracted with the identities

$$B_{jk} = \left[\left(\frac{\partial}{\partial x^j} \lrcorner \varphi\right) \wedge \left(\frac{\partial}{\partial x^k} \lrcorner \varphi\right) \wedge \varphi\right]\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^7}\right), \tag{D.15}$$

$$B_{jk} = -6(g_\varphi)_{jk}\sqrt{\det(g_\varphi)}, \tag{D.16}$$

$$\det B = -(6^7)\det(g_\varphi)^{\frac{9}{2}}, \tag{D.17}$$

$$\sqrt{\det(g_\varphi)} = -\frac{1}{6^{\frac{7}{9}}}(\det B)^{\frac{1}{9}}, \tag{D.18}$$

$$(g_\varphi)_{jk} = \frac{1}{6^{\frac{2}{9}}}\frac{B_{jk}}{(\det B)^{\frac{1}{9}}}. \tag{D.19}$$

In our setup, we have a Kähler Calabi–Yau threefold $X$ with Kähler form $\omega$ and non-vanishing holomorphic $(3,0)$-form $\Upsilon$. We define a $G_2$-structure $\varphi$ on $M = S^1 \times X$ defined by

$$\varphi = \mathrm{Re}\left(\frac{F}{\|\Upsilon\|_\omega}\Upsilon\right) - G dr \wedge \omega, \tag{D.20}$$

where $F$ is non-vanishing and $G > 0$.

Using the local coordinates $r, x^1, y^1, x^2, y^2, x^3, y^3$ on $M$ and associated complex coordinates $z^i = x^i + \sqrt{-1}y^i$, we have

$$\omega = \sqrt{-1}g_{j\overline{k}}dz^j \wedge d\overline{z}^{\overline{k}} \text{ and } \Upsilon = f dz^1 \wedge dz^2 \wedge dz^3. \tag{D.21}$$

Additionally,

$$\|\Upsilon\|_\omega^2 = \frac{|f|^2}{\det g_{p\overline{q}}} = \frac{1}{2^3}\frac{|f|^2}{\sqrt{\det g_{pq}}}. \tag{D.22}$$

Direct computation shows that

$$\begin{aligned}
&\left(\frac{\partial}{\partial r} \lrcorner \varphi\right) \wedge \left(\frac{\partial}{\partial r} \lrcorner \varphi\right) \wedge \varphi \\
&= (-G\omega) \wedge (-G\omega) \wedge \left[\mathrm{Re}\left(\frac{|F|}{\|\Upsilon\|_\omega}\Upsilon\right) - G dr \wedge \omega\right] \\
&= -G^3 dr \wedge \omega^3 \\
&= -6G^3\sqrt{\det g_{jk}} \cdot dr \wedge dx^1 \wedge dy^1 \wedge \ldots \wedge dx^3 \wedge dy^3.
\end{aligned} \tag{D.23}$$

For $j \in \{1,2,3\}$, let $j_1, j_2$ be such that $(j\, j_1\, j_2)$ is a cyclic permutation of $(1\,2\,3)$. Then by considering type decomposition, we can see that

$$\left(\frac{\partial}{\partial r} \lrcorner \varphi\right) \wedge \left(\frac{\partial}{\partial x^j} \lrcorner \varphi\right) \wedge \varphi$$

$$= \underbrace{(-G\omega)}_{(1,1)} \wedge \Big( \underbrace{\frac{1}{2}\frac{Ff}{\|\Upsilon\|_\omega} dz^{j_1} \wedge dz^{j_2}}_{(2,0)} + \underbrace{\frac{1}{2}\frac{\overline{Ff}}{\|\Upsilon\|_\omega} d\overline{z}^{\overline{j_1}} \wedge d\overline{z}^{\overline{j_2}}}_{(0,2)}$$
$$+ \underbrace{Gdr \wedge \sqrt{-1}(g_{j\overline{k}} d\overline{z}^{\overline{k}} - g_{k\overline{j}} dz^k)\Big)}_{dr\wedge(1,1)}$$
$$\wedge \Big[ \underbrace{\mathrm{Re}\Big(\frac{F}{\|\Upsilon\|_\omega}\Upsilon\Big)}_{(3,0)\oplus(0,3)} - \underbrace{Gdr \wedge \omega}_{dr\wedge[(1,0)\oplus(0,1)]} \Big] = 0. \tag{D.24}$$

Similarly, we see that

$$\Big(\frac{\partial}{\partial r} \lrcorner \varphi\Big) \wedge \Big(\frac{\partial}{\partial y^j} \lrcorner \varphi\Big) \wedge \varphi = 0. \tag{D.25}$$

If for $k \in \{1, 2, 3\}$ we define $k_1, k_2$ analogously, we can check that

$$\Big(\frac{\partial}{\partial x^j} \lrcorner \varphi\Big) \wedge \Big(\frac{\partial}{\partial x^k} \lrcorner \varphi\Big) \wedge \varphi$$
$$= \Big( \underbrace{\frac{1}{2}\frac{Ff}{\|\Upsilon\|_\omega} dz^{j_1} \wedge dz^{j_2}}_{(2,0)} + \underbrace{\frac{1}{2}\frac{\overline{Ff}}{\|\Upsilon\|_\omega} d\overline{z}^{\overline{j_1}} \wedge d\overline{z}^{\overline{j_2}}}_{(0,2)}$$
$$+ \underbrace{Gdr \wedge \sqrt{-1}(g_{j\overline{p}} d\overline{z}^{\overline{p}} - g_{p\overline{j}} dz^p)\Big)}_{dr\wedge[(1,0)\oplus(0,1)]}$$
$$\wedge \Big( \underbrace{\frac{1}{2}\frac{Ff}{\|\Upsilon\|_\omega} dz^{k_1} \wedge dz^{k_2}}_{(2,0)} + \underbrace{\frac{1}{2}\frac{\overline{Ff}}{\|\Upsilon\|_\omega} d\overline{z}^{\overline{k_1}} \wedge d\overline{z}^{\overline{k_2}}}_{(0,2)}$$
$$+ \underbrace{Gdr \wedge \sqrt{-1}(g_{k\overline{q}} d\overline{z}^{\overline{q}} - g_{q\overline{k}} dz^q)\Big)}_{dr\wedge[(1,0)\oplus(0,1)]} \wedge \Big[ \underbrace{\mathrm{Re}\Big(\frac{F}{\|\Upsilon\|_\omega}\Upsilon\Big)}_{(3,0)\oplus(0,3)} - \underbrace{Gdr \wedge \omega}_{dr\wedge(1,1)} \Big]$$

$$
\begin{aligned}
&= \Big(\frac{1}{2}\frac{Ff}{\|\Upsilon\|_\omega}dz^{j_1}\wedge dz^{j_2}\Big)\wedge\Big(\frac{1}{2}\frac{\overline{Ff}}{\|\Upsilon\|_\omega}d\overline{z}^{\overline{k_1}}\wedge d\overline{z}^{\overline{k_2}}\Big)\wedge(-Gdr\wedge\omega)\\
&\quad+\Big(\frac{1}{2}\frac{Ff}{\|\Upsilon\|_\omega}dz^{j_1}\wedge dz^{j_2}\Big)\wedge(-Gdr\wedge\sqrt{-1}g_{j\overline{k}}dz^{j})\\
&\qquad\wedge\Big(\frac{1}{2}\frac{\overline{Ff}}{\|\Upsilon\|_\omega}d\overline{z}^{\overline{1}}\wedge d\overline{z}^{\overline{2}}\wedge d\overline{z}^{\overline{3}}\Big)\\
&\quad+\Big(\frac{1}{2}\frac{\overline{Ff}}{\|\Upsilon\|_\omega}d\overline{z}^{\overline{j_1}}\wedge d\overline{z}^{\overline{j_2}}\Big)\wedge\Big(\frac{1}{2}\frac{Ff}{\|\Upsilon\|_\omega}dz^{k_1}\wedge dz^{k_2}\Big)\wedge(-Gdr\wedge\omega)\\
&\quad+\Big(\frac{1}{2}\frac{\overline{Ff}}{\|\Upsilon\|_\omega}d\overline{z}^{\overline{j_1}}\wedge d\overline{z}^{\overline{j_2}}\Big)\wedge(Gdr\wedge\sqrt{-1}g_{k\overline{j}}d\overline{z}^{\overline{j}})\\
&\qquad\wedge\Big(\frac{1}{2}\frac{Ff}{\|\Upsilon\|_\omega}dz^{1}\wedge dz^{2}\wedge dz^{3}\Big)\\
&\quad+(Gdr\wedge\sqrt{-1}g_{j\overline{k}}d\overline{z}^{\overline{k}})\wedge\Big(\frac{1}{2}\frac{\overline{Ff}}{\|\Upsilon\|_\omega}d\overline{z}^{\overline{k_1}}\wedge d\overline{z}^{\overline{k_2}}\Big)\\
&\qquad\wedge\Big(\frac{1}{2}\frac{Ff}{\|\Upsilon\|_\omega}dz^{1}\wedge dz^{2}\wedge dz^{3}\Big)\\
&\quad+(-Gdr\wedge\sqrt{-1}g_{k\overline{j}}dz^{k})\wedge\Big(\frac{1}{2}\frac{Ff}{\|\Upsilon\|_\omega}dz^{j_1}\wedge dz^{j_2}\Big)\\
&\qquad\wedge\Big(\frac{1}{2}\frac{\overline{Ff}}{\|\Upsilon\|_\omega}d\overline{z}^{\overline{1}}\wedge d\overline{z}^{\overline{2}}\wedge d\overline{z}^{\overline{3}}\Big)\\
&= -\frac{3}{4}\frac{|F|^2|f|^2}{\|\Upsilon\|_\omega^2}G\sqrt{-1}\big(g_{j\overline{k}}+g_{k\overline{j}}\big)dr\wedge dz^1\wedge dz^2\wedge dz^3\wedge d\overline{z}^{\overline{1}}\wedge d\overline{z}^{\overline{2}}\wedge d\overline{z}^{\overline{3}}\\
&= -6\frac{|F|^2|f|^2}{\|\Upsilon\|_\omega^2}G\big(g_{j\overline{k}}+g_{k\overline{j}}\big)dr\wedge dx^1\wedge dy^1\wedge\ldots\wedge dx^3\wedge dy^3\\
&= -6\frac{|F|^2|f|^2}{\|\Upsilon\|_\omega^2}Gg\Big(\frac{\partial}{\partial x^j},\frac{\partial}{\partial x^k}\Big)dr\wedge dx^1\wedge dy^1\wedge\ldots\wedge dx^3\wedge dy^3. \qquad \text{(D.26)}
\end{aligned}
$$

Similar computations show that

$$
\begin{aligned}
&\Big(\frac{\partial}{\partial y^j}\lrcorner\varphi\Big)\wedge\Big(\frac{\partial}{\partial y^k}\lrcorner\varphi\Big)\wedge\varphi\\
&\quad= -6\frac{|F|^2|f|^2}{\|\Upsilon\|_\omega^2}Gg\Big(\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}\Big)dr\wedge dx^1\wedge dy^1\wedge\ldots\wedge dx^3\wedge dy^3, \qquad \text{(D.27)}
\end{aligned}
$$

$$
\Big(\frac{\partial}{\partial x^j}\lrcorner\varphi\Big)\wedge\Big(\frac{\partial}{\partial y^k}\lrcorner\varphi\Big)\wedge\varphi
$$

$$= -6\frac{|F|^2|f|^2}{\|\Upsilon\|_\omega^2} Gg\Big(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^k}\Big) dr \wedge dx^1 \wedge dy^1 \wedge \ldots \wedge dx^3 \wedge dy^3. \tag{D.28}$$

With respect to the basis $r, x^1, x^2, x^3, y^1, y^2, y^3$, the matrix $B$ is then

$$\begin{aligned} B &= -6G \begin{bmatrix} G^2\sqrt{\det g_{jk}} & 0 & 0 \\ 0 & \frac{|F|^2|f|^2}{\|\Upsilon\|_\omega^2} C & \frac{|F|^2|f|^2}{\|\Upsilon\|_\omega^2} D \\ 0 & -\frac{|F|^2|f|^2}{\|\Upsilon\|_\omega^2} D & \frac{|F|^2|f|^2}{\|\Upsilon\|_\omega^2} C \end{bmatrix} \\ &= -6G\sqrt{\det g_{jk}} \begin{bmatrix} G^2 & 0 & 0 \\ 0 & \frac{|F|^2}{2^3} C & \frac{|F|^2}{2^3} D \\ 0 & -\frac{|F|^2}{2^3} D & \frac{|F|^2}{2^3} C \end{bmatrix} \\ &= -6G\sqrt{\det g_{jk}} \begin{bmatrix} G^2 & 0 \\ 0 & \frac{|F|^2}{2^3} g_{jk} \end{bmatrix}, \end{aligned} \tag{D.29}$$

which has determinant

$$\det B = (-6)^7 \frac{|F|^{12}}{2^{18}} G^9 (\det g_{jk})^{\frac{9}{2}}. \tag{D.30}$$

As such, it follows that $g_\varphi$ is given by

$$\begin{aligned} g_\varphi &= \frac{-6G\sqrt{\det g_{jk}}}{6^{\frac{2}{9}}(-6)^{\frac{7}{9}}|F|^{\frac{4}{3}} G\sqrt{\det g_{jk}}} \begin{bmatrix} G^2 & 0 \\ 0 & \frac{|F|^2}{2^3} g_{jk} \end{bmatrix} \\ &= \begin{bmatrix} 4|F|^{-\frac{4}{3}} G^2 & 0 \\ 0 & \frac{1}{2}|F|^{\frac{2}{3}} g_{jk} \end{bmatrix}, \end{aligned} \tag{D.31}$$

that is

$$g_\varphi = 4|F|^{-\frac{4}{3}} G^2 dr \otimes dr + \frac{1}{2}|F|^{\frac{2}{3}} g_{jk}. \tag{D.32}$$

From this, we can verify the expressions for the other induced structures such as the volume form $\mathrm{vol}_\varphi$ and the Hodge star $\star_\varphi$. A similar argument handles the setting on a contact Calabi–Yau 7-fold.