Long-Time Existence of the Anomaly Flow

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Outline

Non-Kähler Calabi–Yau Geometry

The Setting

Definition 1.1: Calabi–Yau 3-Folds

A Calabi–Yau 3-fold is a complex manifold X of complex dimension 3 with finite fundamental group and trivial canonical bundle.

Remark 1.2: Ricci-Flatness and Yau's Theorem

We do not require X to admit a Kähler metric ω. If it does, then by Yau's theorem, we get a Ricci-flat Kähler metric $\omega_{CY} \in [\omega]$.

Conifold Transitions

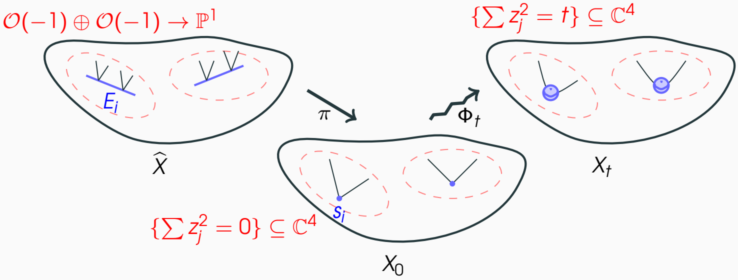

Figure 1: A conifold transition contracts curves on $\widehat{X}$ to points on X_0 and smooths them out to 3-spheres on X_t.

A conifold transition $\widehat{X} \to X_0 \rightsquigarrow X_t$ is a process of deforming one complex compact 3-fold into another.

Locally, it takes neighbourhoods that look like $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$ and applies a blowdown map π, before smoothing out the resulting singularities.

Friedman's Condition

In order to do this globally, we need Friedman's condition.

Theorem 1.3: (R. Friedman '86)

A first-order deformation of X_0 smoothing the singularities $s_i = \pi(E_i)$ exists if and only if there exist $\lambda_i \neq 0$ such that

$$\sum_i \lambda_i [E_i] = 0 \text{ in } H^2(\widehat{X}, \mathbb{R}). \tag{1.1}$$

Kawamata–Tian '92 show that if we have the $\sqrt{-1}\partial\overline{\partial}$-lemma, we get genuine smoothings from the first-order ones.

If we start a conifold transition with a Calabi–Yau 3-fold, the resulting manifolds are still Calabi–Yau.

Reid's Fantasy

Fantasy 1.4: (Reid '87)

All compact Calabi–Yau 3-folds are connected by a sequence of conifold transitions.

This has been verified for large classes of Calabi–Yau 3-folds.
(Candelas–Green–Hübsch '90, Avram–Candelas–Jančić–Mandelberg '96, ...)

Topological Changes

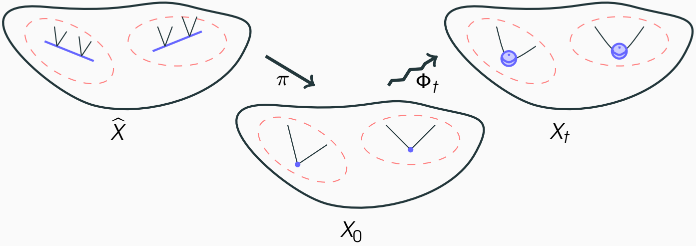

Figure 2: A conifold transition contracts curves on $\widehat{X}$ to points on X_0 and smooths them out to 3-spheres on X_t.

Topologically, a conifold transition contracts 2-cycles from the small resolution and generates 3-cycles on the smoothing.

In particular, if we contract N curves with k linearly independent curves

$$b_2(X_t) = b_2(\widehat{X}) - k, \qquad b_3(X_t) = b_3(\widehat{X}) + 2(N-k). \tag{1.2}$$

A Non-Kähler Example

Let $\widehat{X} = \{\sum_{i=0}^{3} z_i^5 = 0\} \subseteq \mathbb{P}^4$. In this case $\widehat{X}$ is Kähler and $b_2(\widehat{X}) = 1$.

If we pick 2 linearly dependent curves E_1 and E_2 and apply a conifold transition, the resulting smoothing X_t has $b_2(X_t) = 0$.

The Kähler condition is **NOT** preserved by conifold transitions.

This suggests that our main objects of study should include those non-Kähler manifolds obtained from Kähler ones through conifold transitions.

Question 1.5: Model Geometry

What model geometry should we endow these spaces with?

Whatever this geometry is, it should generalize the Ricci-flat Kähler condition.

Compatible Geometries

It is conjectured that the general framework for compact non-Kähler Calabi–Yau geometry should involve not one, but a *pair* of compatible metrics that are unique in appropriate cohomology classes.

The compatibility condition is conjectured to come from the Hull–Strominger system:

Let X be a Calabi–Yau 3-fold and $E \to X$ a holomorphic vector bundle. We have a nowhere vanishing holomorphic $(3,0)$-form Ω.

For a fixed constant $\alpha' \in \mathbb{R}$, we have the Hull–Strominger system which wants a pair of metrics H on E and ω on X such that

$$F_H^{2,0} = F_H^{0,2} = 0, \qquad F_H^{1,1} \wedge \omega^2 = 0, \tag{1.3}$$

$$\sqrt{-1}\partial\overline{\partial}\omega = \alpha' \left(\mathrm{tr}\left(\mathrm{Rm}_\omega \wedge \mathrm{Rm}_\omega \right) - \mathrm{tr}\left(F_H \wedge F_H \right) \right), \tag{1.4}$$

$$d(\|\Omega\|_\omega \omega^2) = 0. \tag{1.5}$$

The curvatures Rm_ω and F_H are Chern curvatures.

The Hull–Strominger System

$$F_H^{2,0} = F_H^{0,2} = 0, \qquad F_H^{1,1} \wedge \omega^2 = 0, \tag{1.3}$$

$$\sqrt{-1}\partial\overline{\partial}\omega = \alpha' \left(\mathrm{tr}\left(\mathrm{Rm}_\omega \wedge \mathrm{Rm}_\omega\right) - \mathrm{tr}\left(F_H \wedge F_H\right)\right), \tag{1.4}$$

$$d(\|\Omega\|_\omega \omega^2) = 0. \tag{1.5}$$

This system of equations arises from heterotic string theory, characterized in terms of SU(3)-structures.

The first equation is a Hermitian Yang–Mills condition between the metrics ω and H.

The second equation is called the heterotic Bianchi identity and comes from the Green–Schwarz anomaly cancellation. The constant α' is called the slope parameter.

The third equation is a conformally balanced condition using the dilaton $\|\Omega\|_\omega$.

Generalizing the Kähler Condition

$$F_H^{2,0} = F_H^{0,2} = 0, \qquad F_H^{1,1} \wedge \omega^2 = 0, \tag{1.3}$$

$$\sqrt{-1}\partial\bar{\partial}\omega = \alpha' \left(\mathrm{tr}\left(\mathrm{Rm}_\omega \wedge \mathrm{Rm}_\omega\right) - \mathrm{tr}\left(F_H \wedge F_H\right)\right), \tag{1.4}$$

$$d(\|\Omega\|_\omega \omega^2) = 0. \tag{1.5}$$

If X admits a Ricci-flat Kähler metric ω, we can set $E = T^{1,0}X$, and $H = \omega$. Doing this solves the Hull–Strominger system.

From this, we can consider solving the Hull–Strominger system as a generalization of the Ricci-flat Kähler condition.

Geometrizing Conifold Transitions

Local Models

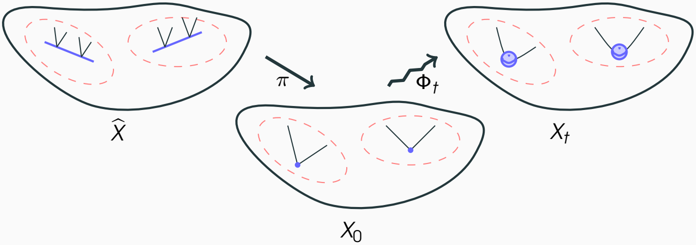

Figure 3: A conifold transition contracts curves on $\widehat{X}$ to points on X_0 and smooths them out to 3-spheres on X_t.

The local neighbourhoods involved in a conifold transition are:

the small resolution $\widehat{V} = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$;

the cone $V_0 = \{z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\} \subseteq \mathbb{C}^4$;

the smoothing $V_t = \{z_1^2 + z_2^2 + z_3^2 + z_4^2 = t\} \subseteq \mathbb{C}^4$.

Local Metrics

Candelas–de la Ossa '90 constructed Ricci-flat Kähler metrics on the local models.

This resulted in metrics $\widehat{\omega}_{co,a}$ on $\widehat{V}$ and metrics $\omega_{co,t}$ on each V_t that are asymptotically conical and have a nice scaling property.

The metric $\omega_{co,0}$ on V_0 is the cone metric

$$dr \otimes dr + r^2 \cdot g_{S^2 \times S^3}. \tag{2.1}$$

These were constructed by using an Ansatz involving an appropriate "radius" function on the local models and imposing the Ricci-flat condition.

Balanced Metrics

Suppose we start a conifold transition $\widehat{X} \to X_0 \rightsquigarrow X_t$ with a Kähler Calabi–Yau 3-fold $\widehat{X}$ with Kähler metric $\widehat{\omega}$. Then by Yau's theorem we have a Calabi–Yau metric $\widehat{\omega}_{CY} \in [\widehat{\omega}]$.

Fu–Li–Yau '12 used this metric and constructed balanced (non-Kähler) metrics $\widehat{\omega}_{FLY,a}$ on $\widehat{X}$ via a gluing process.

A similar process can be done on the smoothings X_t to get balanced metrics $\omega_{FLY,t}$, however this requires an additional pullback and perturbation step as the X_t are distinct spaces.

These metrics are locally modelled on the Candelas–de la Ossa metrics and converge uniformly to the metric $\omega_{FLY,0}$ on the conifold.

Hermitian Yang–Mills Metrics

In the case where $\widehat{X}$ is also simply connected, Collins–Picard–Yau ‘24 constructed new metrics $\widehat{H}_a$ on $\widehat{X}$ and H_t on X_t that are Hermitian Yang–Mills with respect to the Fu–Li–Yau metrics.

These were constructed using a stability argument and an analog of the Donaldson–Uhlenbeck–Yau theorem, starting with the original Calabi–Yau metric $\widehat{\omega}_{CY}$.

Taking a limiting metric and applying a similar gluing, pullback, and perturbation process yields the desired metrics.

Conifold Transitions are Continuous

Since conifold transitions pass through singular spaces and cause jumps in Betti numbers, these allow us to traverse the moduli space of compact Calabi–Yau 3-folds.

We expect this to be continuous in some sense.

Theorem 2.1: (B. Friedman–Picard–S. '24)

Let $\widehat{X}$ be a compact Kähler Calabi–Yau 3-fold and let $\widehat{X} \to X_0 \rightsquigarrow X_t$ be a conifold transition. The geometries $(\widehat{X}, \widehat{g}_{FLY,a}, \widehat{H}_a)$ and $(X_t, g_{FLY,t}, H_t)$ vary continuously in the Gromov–Hausdorff sense and

$$
\begin{aligned}
(\widehat{X}, \widehat{g}_{FLY,a}) &\to (X_0, d_{g_0}) \leftarrow (X_t, g_{FLY,t}) \\
(\widehat{X}, \widehat{H}_a) &\to (X_0, d_{H_0}) \leftarrow (X_t, H_t)
\end{aligned} \tag{2.2}
$$

as $a, t \to 0$ in the Gromov–Hausdorff topology.

The proof involves measuring lengths of curves using the various metrics. This was done by splitting regions close to the exceptional sets/singularities into two regions and comparing them with model spaces with reference metrics.

Anomaly Cancellation

The metrics constructed so far only partially solve the Hull–Strominger system.

$$F_H^{2,0} = F_H^{0,2} = 0, \qquad F_H^{1,1} \wedge \omega^2 = 0, \tag{1.3}$$

$$\sqrt{-1}\partial\bar{\partial}\omega = \alpha' \left(\mathrm{tr}\left(\mathrm{Rm}_\omega \wedge \mathrm{Rm}_\omega\right) - \mathrm{tr}\left(F_H \wedge F_H\right)\right), \tag{1.4}$$

$$d(\|\Omega\|_\omega \omega^2) = 0. \tag{1.5}$$

The Fu–Li–Yau metrics are balanced and the Collins–Picard–Yau metrics satisfy the Hermitian Yang–Mills property, but these together do not solve the heterotic Bianchi identity.

It is expected that full solutions are close to these and can be achieved through perturbative methods.

Anomaly Flow and Long-Time Existence

Anomaly Flow

In order to solve the Hull–Strominger system, Phong–Picard–Zhang proposed the anomaly flow:

$$\partial_t\left(\|\Omega\|_\omega \omega^2\right) = \sqrt{-1}\partial\overline{\partial}\omega - \alpha'\left(\mathrm{tr}\left(\mathrm{Rm}_\omega \wedge \mathrm{Rm}_\omega\right) - \Phi\right). \tag{3.1}$$

Here Φ is a prescribed closed $(2,2)$-form in $c_2(X)$ that can evolve in time.

Chern–Weil theory tells us that the conformally balanced condition is preserved since the RHS is closed.

Coupled Flows

The anomaly flow only evolves the metric ω.

To get full solutions to the Hull–Strominger system, we can couple it with another flow on the metric H on E and prescribe Φ appropriately:

$$H^{-1}\partial_t H = -\Lambda_\omega F_H, \tag{3.2}$$

$$\partial_t\left(\|\Omega\|_\omega \omega^2\right) = \sqrt{-1}\partial\bar{\partial}\omega - \alpha'\left(\mathrm{tr}\left(\mathrm{Rm}_\omega \wedge \mathrm{Rm}_\omega\right) - \mathrm{tr}\left(F_H \wedge F_H\right)\right). \tag{3.3}$$

Properties of the Anomaly Flow

Short-time existence has been shown (Phong–Picard–Zhang `18).

Long-time existence has been shown under certain conditions:

- when $\alpha' = 0$ over Kähler manifolds (Phong–Picard–Zhang `18);
- on T^2-fibrations over $K3$ surfaces (Phong–Picard–Zhang `18);
- on unimodular Lie groups (Phong–Picard–Zhang `19);
- on T^4-fibrations over Riemann surfaces of genus ≥ 2 (Fei–Huang–Picard `21);
- on almost-Abelian Lie groups (using non-Chern connections) (Pujia `21);
- on nilmanifolds (using non-Chern connections) (Pujia–Ugarte `21);

Extending the Anomaly Flow

In the general case, we have the following:

Theorem 3.1: (S. '24)

Suppose that there exist positive constants B, C_0 such that

$$B^{-1} \leq \left(\frac{1}{2\|\Omega\|_\omega}\right) \leq B, \qquad |T|, |\bar{T}|, |\mathrm{Rm}|, |DT|, |D\bar{T}| \leq C_0 \tag{3.4}$$

along the anomaly flow (3.1) on $t \in [0, \tau)$. If α' is sufficiently small, then the flow can be extended to $[0, \tau + \epsilon)$ for some $\epsilon > 0$.

Sketch of Proof

A Hopeful Line of Attack

A general method would ideally follow the method for the $\alpha' = 0$ case:

- Rewrite the flow to get evolution equations for the metric, curvature, and torsion;
- Compute Shi-type estimates for the curvature and torsion;
- Use the maximum principle to get higher regularity for the curvature and torsion;
- Use a bootstrapping argument to show that all derivatives of curvature and torsion are bounded;
- Use that all derivatives are bounded to get long-time existence.

We will try this and highlight difficulties along the way.

Starting Assumptions

We start a bit more generally and assume some higher amount of regularity at first.

Suppose for $k \geq 1$ that there exist positive constants $B, C_0, C_1, \ldots, C_{k-1}$ such that

$$B^{-1} \leq \left(\frac{1}{2\|\Omega\|_\omega}\right) \leq B, \tag{3.5}$$

$$|D^q \mathrm{Rm}|, |D^{q+1} T|, |D^{q+1} \overline{T}| \leq C_q \text{ for } 1 \leq q \leq k-1, \tag{3.6}$$

$$|T|, |\overline{T}|, |\mathrm{Rm}|, |DT|, |D\overline{T}| \leq C_0, \tag{3.7}$$

along the anomaly flow on $t \in [0, \tau)$.

Remark 3.2: Bars

The barred quantities do not matter as much since the Ricci identity and other commutator identities can be used to estimate them from the unbarred ones (at the cost of another constant).

Suppose also that Φ always has enough regularity (or for simplicity, assume $\Phi = 0$)

Evolution Equations

The first step has already been done.

Theorem 3.3: (Phong–Picard–Zhang '18)

Under the anomaly flow (3.1), the metric, curvature, and torsion evolve by

$$\partial_t g = \left(\frac{1}{2\|\Omega\|_\omega}\right)\left[\mathrm{Rm} + T * \overline{T} + \alpha'(\mathrm{Rm} * \mathrm{Rm} + \Phi)\right], \tag{3.8}$$

$$\partial_t \mathrm{Rm} = \left(\frac{1}{2\|\Omega\|_\omega}\right)\left[\frac{1}{2}\Delta_R \mathrm{Rm} + H_1 + \alpha'(\nabla\overline{\nabla}(\mathrm{Rm} * \mathrm{Rm}) + H_2)\right], \tag{3.9}$$

$$\partial_t T = \left(\frac{1}{2\|\Omega\|_\omega}\right)\left[\frac{1}{2}\Delta_R T + K_1 + \alpha'(\nabla(\mathrm{Rm} * \mathrm{Rm}) + K_2)\right]. \tag{3.10}$$

Here H_1, H_2 have at most 2 derivatives of T and $\overline{T}$ and 1 derivative on Rm. Similarly, K_1, K_2 have at most 1 derivatives of T and $\overline{T}$ and no derivatives on Rm.

We can use these to get evolution equations $\partial_t |D^k \mathrm{Rm}|^2$ and $\partial_t |D^{k+1} T|^2$ for norms of derivatives of curvature and torsion.

Pointwise Estimates

After many applications of the CBS inequality and Young's inequality, we get some pointwise estimates.

In particular, for $k \geq 2$ we can define a test function

$$G_k = |D^k \mathrm{Rm}|^2 + |D^{k+1} T|^2 \tag{3.11}$$

and get

$$\begin{aligned}
\partial_t G_k \leq\; & \frac{1}{2}\Big(\frac{1}{2\|\Omega\|_\omega}\Big)\Delta_R G_k - B^{-1} G_{k+1} \\
& + \sum_{m+l=k} 2Re\left(\overline{\nabla}_{\bar{j}}\Big\langle\Big(\frac{\alpha'}{2\|\Omega\|_\omega}\Big)\nabla^{m+1}\overline{\nabla}^{l}(\mathrm{Rm} * \mathrm{Rm}), \nabla^{m}\overline{\nabla}^{l}\mathrm{Rm}\Big\rangle^{\bar{j}}\right) \\
& + \sum_{m'+l'=k+1} 2Re\left(\nabla_{i}\Big\langle\Big(\frac{\alpha'}{2\|\Omega\|_\omega}\Big)\nabla^{m'}\overline{\nabla}^{l'}(\mathrm{Rm} * \mathrm{Rm}), \nabla^{m'}\overline{\nabla}^{l'} T\Big\rangle^{i}\right) \\
& + C\epsilon^{-1}(1 + G_k) + \left[C\epsilon + 6a_0 B C_0 \alpha'\right] G_{k+1}.
\end{aligned} \tag{3.12}$$

The terms in red are higher-order but non-Laplacian and so we cannot appeal to the maximum principle to deal with them.

Integral Estimates

To deal with the extra terms, we integrate and use L^p-norms instead.

By integrating, we get that

$$
\begin{aligned}
&\partial_t\Big(\int_X G_k^p\Big)\\
&\leq \frac{p}{2}\int_X\Big(\frac{1}{2\|\Omega\|_\omega}\Big)G_k^{p-1}\cdot(\Delta_R G_k) - B^{-1}p\int_X G_k^{p-1}\cdot G_{k+1}\\
&\quad+\sum_{m+l=k} 2pRe\Big(\int_X G_k^{p-1}\cdot\overline{\nabla}_{\bar{j}}\Big\langle\Big(\frac{\alpha'}{2\|\Omega\|_\omega}\Big)\nabla^{m+1}\overline{\nabla}^{l}(\mathrm{Rm}*\mathrm{Rm}),\nabla^{m}\overline{\nabla}^{l}\mathrm{Rm}\Big\rangle^{\bar{j}}\Big)\\
&\quad+\sum_{m'+l'=k+1} 2pRe\Big(\int_X G_k^{p-1}\cdot\nabla_{i}\Big\langle\Big(\frac{\alpha'}{2\|\Omega\|_\omega}\Big)\nabla^{m'}\overline{\nabla}^{l'}(\mathrm{Rm}*\mathrm{Rm}),\nabla^{m'}\overline{\nabla}^{l'}T\Big\rangle^{i}\Big)\\
&\quad+C\epsilon^{-1}\int_X G_k^{p-1}\cdot(1+G_k)+\big[C\epsilon+6a_0BC_0\alpha' p\big]\int_X G_k^p\cdot G_{k+1}.
\end{aligned}
\tag{3.13}
$$

Integration by Parts

Applying the Divergence Theorem and rearranging, this gives

$$\partial_t\Big(\int_X G_k^p\Big) \leq C\epsilon^{-1}\int_X (1+G_k^p)$$
$$+ \big[C\epsilon + 4a_0BC_0\alpha' p(p+\frac{1}{2}) - B^{-1}p\big]\int_X G_k^{p-1}\cdot G_{k+1}$$
$$+ \big[C\epsilon + 4a_0BC_0\alpha' p(p-1) - B^{-1}p(p-1)\big]\int_X G_k^{p-2}\cdot|\overline{\nabla} G_k|^2. \tag{3.14}$$

This mean that if

$$\alpha' < \frac{1}{4a_0B^2C_0(p+\frac{1}{2})}, \tag{3.15}$$

then we can pick $\epsilon = \epsilon(k,\alpha',p)$ such that the blue terms are negative.

This leaves us with

$$\partial_t\Big(\int_X G_k^p\Big) \leq C + C\int_X G_k^p, \tag{3.16}$$

for some constant $C = C(k,\alpha',p)$.

Proposition 3.4: Grönwall's Inequality

Let β and u be real-valued continuous functions defined on an interval $[a, b)$. If u is differentiable on (a, b) with

$$u'(t) \leq \beta(t) \cdot u(t), \qquad t \in (a, b), \tag{3.17}$$

then

$$u(t) \leq u(a) \exp\Big(\int_a^t \beta(s) ds\Big), \qquad t \in [a, b). \tag{3.18}$$

Applying this to the functions

$$u = 1 + \int_X G_k^p, \qquad \beta = C, \tag{3.19}$$

we conclude that

$$\int_X G_k^p(t) \leq \Big(1 + \int_X G_k^p(0)\Big) e^{Ct} < \Big(1 + \int_X G_k^p(0)\Big) e^{C\tau}. \tag{3.20}$$

Overall, we see that if α' is sufficiently small, then

$$\int_X G_k^p = \int_X \left(|D^k \mathrm{Rm}|^2 + |D^{k+1} T|^2\right)^p \tag{3.21}$$

is uniformly bounded along the flow.

Importantly, this bound does not depend on k for bootstrapping later.

By taking a $2p$-th root, we see that

$$|D^k \mathrm{Rm}| \text{ and } |D^{k+1} T| \tag{3.22}$$

are uniformly L^{2p}-bounded.

Remark 3.5: "p-Values"

This argument only works for $p \geq 3$, but since X is compact, we can use Hölder's inequality to get this for $1 \leq p < 3$.

Reobtaining Pointwise Estimates

We now have uniform L^{2p}-bounds for $|D^k \mathrm{Rm}|$ and $|D^{k+1} T|$, but we need to get pointwise ones back.

To do this, we appeal to an argument of Hamilton '82 and use the Sobolev Embedding Theorem.

If we get uniform L^{2p}-bounds on $|D^{k+1} \mathrm{Rm}|$ and $|D^{k+2} T|$ for large enough p, then the Sobolev Embedding Theorem will give uniform L^{∞}-bounds for $|D^k \mathrm{Rm}|$ and $|D^{k+1} T|$ along the flow.

Higher-Order Estimates

We can repeat the previous steps for $k \geq 3$ with the test function

$$G_{k+1} = |D^{k+1}\mathrm{Rm}|^2 + |D^{k+2}T|^2. \tag{3.23}$$

This gives

$$\begin{aligned}\partial_t\Big(\int_X G_{k+1}^p\Big) \leq C(\epsilon')^{-1}\int_X (1 + G_k^p + G_{k+1}^p) \\ + [C\epsilon' + 4a_0BC_0\alpha' p(p+\frac{1}{2}) - B^{-1}p]\int_X G_{k+1}^{p-1}\cdot G_{k+2} \\ + [C\epsilon' + 4a_0BC_0\alpha' p(p-1) - B^{-1}p(p-1)]\int_X G_{k+1}^{p-2}\cdot|\overline{\nabla}G_{k+1}|^2.\end{aligned} \tag{3.24}$$

The Grönwall's Inequality argument gives the same condition on α':

$$\alpha' < \frac{1}{4a_0B^2C_0(p+\frac{1}{2})}. \tag{3.25}$$

If this holds, we see that there exists some positive constant C_k such that

$$|D^k\mathrm{Rm}|, |D^{k+1}T|, |D^{k+1}\overline{T}| \leq C_k. \tag{3.26}$$

Bootstrapping

Using a bootstrapping argument, we get

Proposition 3.6: $k = 3$

Suppose our starting assumptions hold for $k = 3$. If

$$\alpha' < \frac{1}{14 a_0 B^2 C_0}, \qquad (p = 3 \text{ is sufficient}) \tag{3.27}$$

then there exist positive constants C_q for $q \geq 3$ such that

$$|D^q \mathrm{Rm}|, |D^{q+1} T|, |D^{q+1} \overline{T}| \leq C_q \tag{3.28}$$

along the anomaly flow on $[0, \tau)$.

Additional Complications

When k is small, some terms gain some extra higher-order dependence in their bounds.

For example, if $k = 2$, then the terms

$$|D^2\mathrm{Rm}|, |D^3\mathrm{Rm}|, |D^4\mathrm{Rm}| \tag{3.29}$$

are a priori unknown and have no bounds.

Our earlier method tried to obtain terms at most quadratic in these unknowns. In this case, however

$$\langle D^4(\mathrm{Rm} * \mathrm{Rm}), D^4\mathrm{Rm}\rangle \leq C|D^4\mathrm{Rm}|^2 + C|D^3\mathrm{Rm}| \cdot |D^4\mathrm{Rm}| + C|D^2\mathrm{Rm}|^2 \cdot |D^4\mathrm{Rm}|. \tag{3.30}$$

The $k = 2$ Case

This case is not too bad because we have the estimates on G_k as a start. (We only needed $k \geq 3$ for the estimates on G_{k+1}).

The main difference here is that instead of

$$\partial_t\Big(\int_X G_{k+1}^p\Big) \leq C(\epsilon')^{-1}\int_X (1 + G_k^p + G_{k+1}^p) + \big[C\epsilon' + 4a_0BC_0\alpha' p(p+\frac{1}{2}) - B^{-1}p\big]\int_X G_{k+1}^{p-1}\cdot G_{k+2} + \big[C\epsilon' + 4a_0BC_0\alpha' p(p-1) - B^{-1}p(p-1)\big]\int_X G_{k+1}^{p-2}\cdot|\overline{\nabla}G_{k+1}|^2. \tag{3.31}$$

we get

$$\partial_t\Big(\int_X G_3^p\Big) \leq C(\epsilon')^{-1}\int_X (1 + G_2^p + G_2^{2p} + G_3^p) + \big[C\epsilon' + 4a_0BC_0\alpha' p(p+\frac{1}{2}) - B^{-1}p\big]\int_X G_3^{p-1}\cdot G_4 + \big[C\epsilon' + 4a_0BC_0\alpha' p(p-1) - B^{-1}p(p-1)\big]\int_X G_3^{p-2}\cdot|\overline{\nabla}G_3|^2. \tag{3.32}$$

A Slightly Tighter Bound

To compensate for the G_2^{2p} appearing, we need to adjust our condition on α' to ensure that $\int_X G_2^{2p}$ is uniformly bounded:

$$\alpha' < \frac{1}{4a_0 B^2 C_0 (2p + \frac{1}{2})} \tag{3.33}$$

This improves our result to

Proposition 3.7: $k = 2$

Suppose our starting assumptions hold for $k = 2$. If

$$\alpha' < \frac{1}{26 a_0 B^2 C_0}, \qquad (p = 3 \text{ is sufficient}) \tag{3.34}$$

then there exist positive constants C_q for $q \geq 2$ such that

$$|D^q \mathrm{Rm}|, |D^{q+1} T|, |D^{q+1} \overline{T}| \leq C_q \tag{3.35}$$

along the anomaly flow on $[0, \tau)$.

The $k = 1$ Case

This case is more complicated since we need to reprove the estimates on G_k.

We actually need a more complicated test function

$$G = [\alpha'(|\mathrm{Rm}|^2 + |DT|^2) + \mu] \cdot (|D\mathrm{Rm}|^2 + |D^2 T|^2) = [\alpha' G_0 + \mu] \cdot G_1. \qquad (3.36)$$

Here, the constant μ is to be determined later.

“Fun” Conditions

Going through the steps, we eventually get

$$
\begin{aligned}
&\partial_t\Big(\int_X G^p\Big)\\
&\leq C\epsilon^{-1}\int_X (1+G^p)\\
&\quad + \Big[C\epsilon + 6a_0BC_0^2(\alpha')^2\mu^{-1}p(p-1) + 10a_0B\alpha' p(p-1)\\
&\qquad - B^{-1}p(p-1)\Big]\int_X G^{p-2}\cdot|\overline{\nabla}G|^2\\
&\quad + \Big[C\epsilon + 140a_0BC_0^2(\alpha')^2p + 26a_0BC_0^2(\alpha')^2p(p-1) + 86a_0BC_0(\alpha')^2p\\
&\qquad + 10a_0B\alpha'\mu p^2 + \frac{1}{2}B^{-1}\alpha' p - B^{-1}\alpha' p\Big]\int_X G^{p-1}\cdot G_1^2\\
&\quad + \Big[C\epsilon + 20a_0BC_0^4(\alpha')^2p(p-1) + 8320B^3C_0^2\alpha' p + 12a_0BC_0^3(\alpha')^2p\\
&\qquad + 128a_0BC_0^2(\alpha')^2p + 10a_0BC_0^2\alpha'\mu p(p-1)\\
&\qquad + 6a_0BC_0\alpha'\mu p + 4a_0B\alpha'\mu p - B^{-1}\mu p\Big]\int_X G^{p-1}\cdot G_2.
\end{aligned}
\tag{3.37}
$$

New Bounds

To make the blue terms negative, we need to solve the system which works when

$$\alpha' < \frac{1}{10^6 a_0 B^6 \max(1, C_0)^2 p}, \qquad \mu = \frac{1}{100 a_0 B^2 p}. \tag{3.38}$$

To get the higher-order estimates, we follow the same method for the test function

$$G' = \left[\alpha' \left(|\mathrm{Rm}|^2 + |DT|^2\right) + \mu'\right] \cdot \left(|D^2\mathrm{Rm}|^2 + |D^3 T|^2\right) = \left[\alpha' G_0 + \mu'\right] \cdot G_2. \tag{3.39}$$

A similar system appears and can be solved if

$$\alpha' < \frac{1}{10^7 a_0 B^6 \max(1, C_0)^2 p}, \qquad \mu = \frac{1}{100 a_0 B^2 p}. \tag{3.40}$$

The Final Upgrade

From this, we can again upgrade our proposition

Proposition 3.8: $k = 1$

Suppose our starting assumptions hold for $k = 1$. If

$$\alpha' < \frac{1}{3 \cdot 10^7 a_0 B^6 \max(1, C_0)^2}, \qquad (p = 3 \text{ is sufficient}) \tag{3.41}$$

then there exist positive constants C_q for $q \geq 1$ such that

$$|D^q \mathrm{Rm}|, |D^{q+1} T|, |D^{q+1} \overline{T}| \leq C_q \tag{3.42}$$

along the anomaly flow on $[0, \tau)$.

Now that all derivatives are uniformly bounded, we can now invoke the argument of Phong–Picard–Zhang `18 to extend the flow.

Remark 3.9: Dimensionality

The bounds we get for the $k = 1$ and the $k \geq 2$ cases are interesting since they can be rewritten as

$$\alpha' \cdot |\mathrm{Rm}|^2 < \Pi_1, \qquad (k = 1), \tag{3.43}$$

$$\alpha' \cdot |\mathrm{Rm}| < \Pi_2, \qquad (k \geq 2) \tag{3.44}$$

for dimensionless constants Π_1, Π_2. The units on the LHS of each differ but both RHS are dimensionless.

Question 3.10: Rescaling

Are there rescaling methods to go further with this result?

Question 3.11: Coupling Flows

Can this be done while coupling the anomaly flow with another flow on H and setting $\Phi = \mathrm{tr}\,(F_H \wedge F_H)$?

Fin

Thank you for your attention.

Questions?