Research Statement Caleb Suan

1 Introduction

My research interests lie in differential geometry and geometric analysis. In particular, I focus on geometric flows on manifolds with special holonomy. These have proven to be a useful tool in solving problems in geometry and topology; the Ricci flow, for example, was key in Perelman's proof of the Poincaré conjecture and Thurston's geometrization conjecture [31, 32, 33].

My work has mostly been in the setting of Calabi–Yau 3-folds and G_2 manifolds, which arise as special cases of Berger's classification. A lot has been done in studying torsion-free structures, however there is also interest in structures with torsion – an example being solutions to the Hull–Strominger system. This system of PDEs from string theory which was proposed in the 1980's has produced a mathematically rich and interesting setting to study. It is conjectured[1] [43] that all compact Calabi–Yau 3-folds (complex 3-folds with finite fundamental group and trivial canonical bundle) can be deformed into each other by iterating a process known as a conifold transition. It is further speculated that these manifolds all admit solutions to the Hull–Strominger system and that these solutions provide the model geometry in this setting.

2 Overview of Research Projects

Below is an overview of my past, current, and future research projects.

Past Projects

I. (Friedman–Picard–S. [16]) In joint work with B. Friedman and S. Picard, we showed that conifold transitions are continuous in the Gromov–Hausdorff topology when using metrics constructed by Fu–Li–Yau [19] and Collins–Picard–Yau [12]. (§3.1)

II. (S. [45]) I proved a sufficient smallness condition on the slope parameter α' of the anomaly flow on $[0, \tau)$ which allows the flow to be extended to $[0, \tau + \epsilon)$. (§3.2)

III. (Picard–S. [38]) Together with S. Picard, we proved that the Laplacian flow and coflow on a trivial S^1-bundle over a Calabi–Yau 3-fold reduce to Monge–Ampère flows on the base. (§4.1)

IV. (Sá Earp–Saavedra–S. [46]) In joint work with H. Sá Earp and J. Saavedra, we extended results from our previous work [29, 38] by considering S^1-reductions of the (modified) Laplacian coflow on contact Calabi–Yau 7-folds, giving necessary and sufficient conditions for certain Ansätze to satisfy the flows. (§4.1)

Current and Future Projects

I. Desingularisation of Non-Kähler Calabi–Yau Conifolds. (§3.1, Problem 3.3)

II. Long-time Existence of Flows with Non-Laplacian Higher-Order Terms. (§3.2, Problem 3.5)

III. Surgery Techniques for the Anomaly Flow. (§3.2, Problem 3.6)

IV. Properties of Modified G_2 Anomaly Flows and Laplacian Coflows. (§4.2, Problem 4.3)

In the following sections, I provide some brief background on my research projects and explain some of the main results, while describing avenues for new research projects.

3 Non-Kähler Calabi–Yau Geometry

A conifold transition – denoted $\widehat{X} \to X_0 \rightsquigarrow X_t$ – is a process that deforms a compact Calabi–Yau 3-fold $\widehat{X}$ into a family X_t of such spaces while passing through an intermediate space X_0 with cone singularities (*i.e.,* a conifold). Locally, it contracts smooth rational curves on $\widehat{X}$ to points via a blowdown, and subsequently smooths out the resulting singularities with 3-spheres (see Figure 1). Results in [17, 18, 27, 42, 47] provide homological conditions on the curves that determine when this process can be achieved globally.

Topologically, conifold transitions contract 2-cycles and replace them with 3-cycles, which correspond to changes in Betti numbers. In particular, b_2 decreases while b_3 increases. Consequently, the Kähler property

[1] one might even say "fantasised"

is not necessarily preserved by this process. As an example, consider the quintic $\widehat{X} = \{\sum_{i=0}^{3} z_i^5 = 0\} \subseteq \mathbb{CP}^4$. This space has $b_2(\widehat{X}) = 1$ and by contracting a pair of curves, we get spaces X_t with $b_2(X_t) = 0$ and hence cannot admit Kähler metrics.

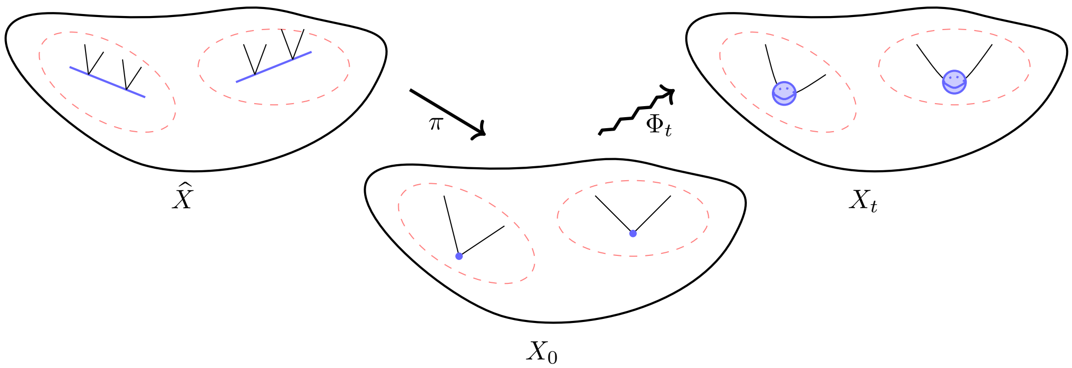

Figure 1: A conifold transition contracts curves on $\widehat{X}$ to points on X_0 and smooths them out to 3-spheres on X_t.

Reid's fantasy conjectures that all Calabi–Yau 3-folds can be linked by a sequence of conifold transitions [43] and has since been verified for large classes of examples [2, 6]. Since the Kähler condition is not preserved, this suggests that a Ricci-flat Kähler metric is not the "correct" geometry to endow these spaces with. A central open problem in the field is to understand what the right generalisation should be. A conjecture of Yau says that the model geometry should instead be a *pair* of compatible metrics and that the relevant compatibility comes from the Hull–Strominger system:

Conjecture 3.1 (Reid [43], Yau)**.** All Calabi–Yau 3-folds can be linked by a sequence of conifold transitions. Further, each of these spaces admits a unique solution to the Hull–Strominger system (3.1) - (3.3) in a suitable cohomology class.

Let X be a compact complex 3-fold with holomorphic volume form Υ and a holomorphic vector bundle $E \to X$. The Hull–Strominger system looks for a Hermitian metric ω on X and a Hermitian metric H on E that satisfy

$$F^{2,0} = F^{0,2} = 0, \qquad \omega \wedge F^{1,1} = 0, \tag{3.1}$$

$$\sqrt{-1}\partial\overline{\partial}\omega - \alpha'\Big(\mathrm{tr}\,(\mathrm{Rm} \wedge \mathrm{Rm}) - \mathrm{tr}\,(F \wedge F)\Big) = 0, \tag{3.2}$$

$$d(\|\Upsilon\|_\omega \omega^2) = 0. \tag{3.3}$$

where Rm and F are the Chern curvatures of ω and H, respectively, and α' is a given constant called the slope parameter which tunes the behaviour of the system.

These equations, proposed in [23, 44], generalise the compactification of the 10D heterotic string in [7]. The first condition is a Hermitian Yang–Mills relation between the metrics ω and H. Condition (3.2) is called the heterotic Bianchi identity and comes from the Green–Schwarz anomaly cancellation [21]. The final condition is that ω is conformally balanced with respect to the dilaton $\|\Upsilon\|_\omega$.

The prototypical example of a solution is that of a Ricci-flat Kähler metric: If ω is a Ricci-flat Kähler metric on X, then setting $E = T^{1,0}X$ and $H = \omega$ yields a solution to the Hull–Strominger system since ω is closed and the norm $\|\Upsilon\|_\omega$ is constant, hence justifying this system as a generalisation of the Ricci-flat Kähler condition.

3.1 Gromov–Hausdorff Continuity of Conifold Transitions

There has been much partial progress on finding solutions to the Hull–Strominger system through conifold transitions $\widehat{X} \to X_0 \rightsquigarrow X_t$. Fu–Li–Yau [19] used a gluing method to construct balanced (non-Kähler) metrics $\widehat{g}_{FLY,a}$, $g_{FLY,t}$ on $\widehat{X}$ and X_t respectively under the assumption that the initial manifold $\widehat{X}$ is Kähler. These were based on local models studied by Candelas–de la Ossa [5]. More recently, Collins–Picard–Yau [12] constructed metrics $\widehat{H}_a$, H_t on the respective tangent bundles satisfying the Hermitian Yang–Mills condition (3.1) with respect to the metrics in [19].

As conifold transitions pass through intermediate singular spaces and cause jumps in Betti numbers, they allow us to traverse the moduli space of Calabi–Yau 3-folds. As such, we expect this process to be continuous in some sense. In joint work with B. Friedman and S. Picard, we made this rigorous by showing that a conifold transition is continuous in the Gromov–Hausdorff topology when endowed with the metrics in [12, 19].

Theorem 3.2 (Friedman–Picard–S. [16]). *Let $\widehat{X}$ be a compact Kähler Calabi–Yau 3-fold and let $\widehat{X} \to X_0 \rightsquigarrow X_t$ be a conifold transition. The geometries $(\widehat{X}, \widehat{g}_{FLY,a}, \widehat{H}_a)$ and $(X_t, g_{FLY,t}, H_t)$ vary continuously in the Gromov–Hausdorff sense and*

$$\begin{aligned} (\widehat{X}, \widehat{g}_{FLY,a}) &\to (X_0, d_{g_0}) \leftarrow (X_t, g_{FLY,t}) \\ (\widehat{X}, \widehat{H}_a) &\to (X_0, d_{H_0}) \leftarrow (X_t, H_t) \end{aligned} \tag{3.4}$$

as $a, t \to 0$ in the Gromov–Hausdorff topology.

The proof involved measuring lengths of curves between points using different metrics. Neighbourhoods around each exceptional set/singularity were partitioned into two parts: an inner "tube" (or "disc") and a surrounding "annulus". By comparing each of these sets to model spaces endowed with reference metrics, we showed that the variation in distances caused by a conifold transition is small.

The metrics in [12, 19] only partially solve the Hull–Strominger system as they fail to satisfy the heterotic Bianchi identity (3.2). Even though this is the case, Theorem 3.2 can be considered as evidence to the affirmative of Conjecture 3.1 as it is expected that solutions to the full system can be obtained by perturbing the chosen metrics. If such solutions can be found, then our result could be used to show that conifold transitions are still continuous with respect to the new metrics as the proof is amenable to small perturbations.

A possible start to proving Conjecture 3.1 is to adapt the desingularisation methods of [9, 10, 25, 28] to solutions of the Hull–Strominger system on conifolds. In each case, (modified) results of Joyce [24] were needed to perturb the structures constructed by the gluing process to ones with the desired torsion properties. If an analogous result holds for solutions of the Hull–Strominger system, one could possibly obtain solutions of the system through the conifold transition process.

Problem 3.3. Adapt the gluing constructions of [9, 10, 25, 28] to the setting of non-Kähler Calabi–Yau conifold transitions and obtain an analogue of Joyce's results [24] for the Hull–Strominger system to obtain solutions to the system.

3.2 Long-Time Existence of the Anomaly Flow

Another method of studying the Hull–Strominger system is through the use of geometric flows. One such flow is the anomaly flow, proposed in [36], and given by the equation and initial condition

$$\begin{cases} \frac{\partial}{\partial t}(\|\Upsilon\|_\omega \omega^2) = \sqrt{-1}\partial\overline{\partial}\omega - \alpha'\Big(\mathrm{tr}\,(\mathrm{Rm} \wedge \mathrm{Rm}) - \Phi\Big), \\ d(\|\Upsilon\|_{\omega_0}\omega_0^2) = 0. \end{cases} \tag{3.5}$$

Here Φ is a prescribed closed $(2,2)$-form in $c_2(X)$. By design, it preserves the conformally balanced condition (3.3) and can be coupled with a flow on the metric H to yield full solutions to the system.

In [36], short-time existence is proven and long-time existence results have been shown in various settings [14, 34, 35, 37, 41, 40]. Adding to this list, I showed that the anomaly flow can be extended under a condition on the slope parameter α':

Theorem 3.4 (S. [45]). *Suppose that the dilaton $\|\Upsilon\|_\omega$, curvature Rm, torsion T, and its first derivatives DT are bounded along the anomaly flow (3.5) on $[0,\tau)$. If α' is sufficiently small then the flow can be extended to $[0,\tau+\epsilon)$ for some $\epsilon > 0$.*

The proof involved computing Shi-type estimates for the anomaly flow. A distinguishing feature of the anomaly flow is a non-Laplacian higher-order term $\alpha'\big(\nabla\overline{\nabla}(\mathrm{Rm}*\mathrm{Rm})\big)$ in the evolution $\frac{\partial}{\partial t}\mathrm{Rm}$ of the curvature. This added a new difficulty in obtaining the estimates as the extra term is not amenable to the usual maximum principle techniques. To circumvent this, I used integral estimates and an integration-by-parts method to lower the order of the extra term by one, before employing the Sobolev embedding theorem to reobtain pointwise estimates. Another example of using integral estimates in studying flows can be seen in [11].

To my knowledge, the integration-by-parts technique employed in the proof of Theorem 3.4 was the first to handle this sort of extra term. This term generally occurs for geometric flows when the evolution $\frac{\partial}{\partial t} g$ of the metric involves an $\alpha'(\mathrm{Rm} * \mathrm{Rm})$ term. For example, the 2-loop renormalisation group flow (RG-2 flow) [8, 20] and the heterotic-Ricci flow [30] both have this characteristic. As a related problem, we have the following:

Problem 3.5. Obtain long-time existence results for flows with extra non-Laplacian higher-order terms (such as the RG-2 flow and the heterotic-Ricci flow).

I expect that an adaptation of the integration-by-parts argument should allow us to obtain analogous Shi-type estimates and long-time existence results for these flows, depending on the value of α'.

From the long-time existence result in Theorem 3.4, we have a better understanding of how singularities of the anomaly flow might occur. Taking inspiration from work done with the Ricci flow, we can consider how to circumvent singularity formation using rescaling and "surgery" techniques. This gives another avenue to tackling Conjecture 3.1 since bypassing the singularities of the anomaly flow may allow us to obtain long-time solutions to the Hull–Strominger system.

Problem 3.6. Use the curvature and torsion bounds of [45] to develop rescaling and "surgery" arguments for the anomaly flow.

4 Flows in Dimension 7

A G_2 structure on a 7-fold M is a 3-form φ which satisfies a certain non-degeneracy condition. These occur as an exceptional case of Berger's classification and induce a Riemannian metric g_φ, volume form vol_φ, and Hodge star $\star_\varphi$, and dual 4-form $\psi = \star_\varphi \varphi$.

A natural class of G_2 structures are those that are torsion-free – those φ such that $\nabla_\varphi \varphi = 0$, where ∇_φ is the Levi-Civita connection of the associated metric g_φ, which in turn is determined by φ non-linearly. A theorem of Fernández–Gray [15] says that φ is torsion-free if and only if $d\varphi = d\psi = 0$.

Several geometric flows have been used to try and find such G_2 structures such as the Laplacian flow [3] and the dual Laplacian coflow [26]. These are given respectively by

$$\begin{cases} \frac{\partial}{\partial t}\varphi = \Delta_d \varphi, \\ d\varphi_0 = 0, \end{cases} \quad \text{and} \quad \begin{cases} \frac{\partial}{\partial t}\psi = \Delta_d \psi, \\ d\psi_0 = 0, \end{cases} \tag{4.1}$$

where $\Delta_d = dd_\varphi^* + d_\varphi^* d$ denotes the Hodge Laplacian with respect to the (changing) metric g_φ. From this we can see that both flows preserve the closed condition. The fixed points of both flows are torsion-free G_2 structures even if M is non-compact. Bryant–Xu [4] has showed that the Laplacian flow has short-time existence and uniqueness in the compact case, however these are still unknown for the coflow.

The Bryant–Xu [4] proof of short-time existence and uniqueness for the Laplacian flow does not work for the coflow. To remedy this, Grigorian [22] proposed a modification involving the intrinsic torsion T and a free parameter A:

$$\begin{cases} \frac{\partial}{\partial t}\psi = \Delta_d \psi - 2d\big((\operatorname{tr} T - A)\varphi\big), \\ d\psi_0 = 0, \qquad A \in \mathbb{R}. \end{cases} \tag{4.2}$$

This new coflow has short-time existence and uniqueness. However, the drawback is that fixed points are not necessarily torsion-free.

4.1 S^1-Reduction of G_2 Flows

The inclusion $\mathrm{SU}(3) \subseteq G_2$ suggests an intimate connection between manifolds with such structures. For example, given a Kähler Calabi–Yau 3-fold X with Kähler form ω and holomorphic volume form Υ we can construct a G_2 structure on a $S^1 \times X$ by setting

$$\varphi = \mathrm{Re}\left(\frac{F}{\|\Upsilon\|_\omega}\Upsilon\right) - G dr \wedge \omega. \tag{4.3}$$

Here, F and G are appropriately chosen functions on X and r is the angle coordinate r on S^1.

Using this construction, S. Picard and I considered the Laplacian coflow on the trivial fibration $M = S^1 \times X$ and showed that it reduces to the Kähler-Ricci flow on the base X:

Theorem 4.1 (Picard–S. [38])**.** *In the above setup with $F = 1$ and $G = \|\Upsilon\|_\omega$, if ω_t is a solution to the Kähler–Ricci flow on X, then the construction in (4.3) is a solution to the Laplacian coflow (up to pullback by a family of diffeomorphisms).*

In our paper, we also proved that the Laplacian flow (with the choice of F and G reversed) analogously reduces to the $\mathrm{MA}^{\frac{1}{3}}$ flow, a member of a class of well-behaved flows called Monge–Ampère flows [39]. Similar reductions were shown for trivial T^3-fibrations over Kähler Calabi–Yau surfaces. In addition to reducing these flows to the base X, we showed that they converge to torsion-free structures as $t \to \infty$, giving the first non-perturbative examples of long-time existence and convergence for compact G_2 flows.

In [46], H. Sá Earp, J. Saavedra and I extended the results from [29, 38]. We considered a more general construction for non-trivial S^1-fibrations using contact Calabi–Yau 7-folds. We also applied our methods to the modified coflow on the trivial S^1-fibration and both coflows on contact Calabi–Yau 7-folds.

Here the reduced equations are more complicated, however we still obtained necessary and sufficient conditions for structures on the base to solve the (modified) coflow via a construction similar to (4.3). In addition, we also performed a singularity analysis for the modified coflow on contact Calabi–Yau 7-folds in the case of a particular initial condition:

Proposition 4.2 (Sá Earp–Saavedra–S. [46])**.** *On a compact contact Calabi–Yau 7-fold M with transverse Ricci-flat Kähler form $\omega = d\eta$, the solution to the modified Laplacian coflow with $A = 0$ and initial condition*

$$\varphi_0 = \mathrm{Re}\,\Upsilon + \epsilon\eta \wedge \omega \tag{4.4}$$

has a Type I finite-time singularity at $\tau = \frac{1}{5}\epsilon^{-2}$. If we normalise (M, g_t) to a fixed volume, then it collapses to $\mathbb{R}$, as $t \to \tau$.

4.2 Modified G_2-Anomaly Flows and Laplacian Coflows

The symmetries of the Hull–Strominger system can be generalised to other settings, including manifolds with G_2 structure. In this setting, a G_2 analogue of the anomaly flow was proposed by Ashmore–Minasian–Proto [1]:

$$\begin{cases} \frac{\partial}{\partial t}(e^{-2f}\psi) = d(e^{2f}d^*(e^{-2f}\psi)) - \frac{2}{3}d\big((\mathrm{tr}\,T)\varphi\big) \\ d(e^{-2f_0}\psi_0) = 0 \end{cases} \tag{4.5}$$

Here, the function e^{-2f} takes the role of the dilaton and behaves similarly to $\|\Upsilon\|_\omega$ from (3.5). When f is constant in time, this becomes a particular case of the modified coflow (4.2) which hints toward studying these flows together as part of a larger class of "modified G_2 anomaly flows". We then have the natural questions regarding geometric flows, such as their fixed points, short- and long-time existence, uniqueness, and convergence of solutions.

Problem 4.3. Describe the full space of fixed points of modified G_2 anomaly flows. Further, verify short-time existence and uniqueness of these flows and study their long-time behaviour and convergence.

At the time of writing, I have obtained certain necessary conditions for fixed points using type decomposition of forms. A full classification of the fixed points is work in progress.

It is expected that this flow does have the properties of short-time existence and uniqueness. Currently, S. Karigiannis, S. Picard and I have been able to show these properties for certain modified anomaly flows by adjusting some parameters. We hope to strengthen what we currently have by fitting these flows into the general theory of G_2 flows described by Dwivedi–Gianniotis–Karigiannis [13].

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