CRM - Special Riemannian Geometries in Dimensions $6, 7, 8$

Gromov–Hausdorff Convergence of Non-Kähler Calabi–Yau Conifold Transitions

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Outline

Conifold Transitions

Conifold Transitions

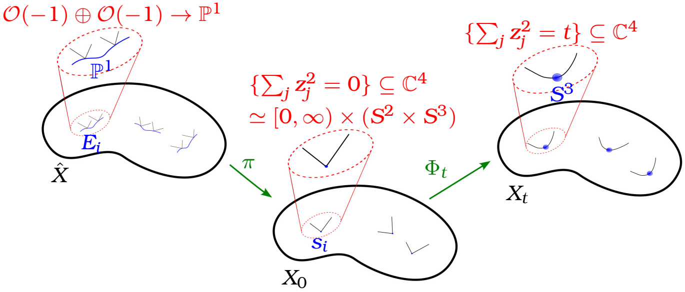

Figure: A Conifold Transtion

A conifold transition $\widehat{X} \to X_0 \rightsquigarrow X_t$ is a process of deforming one complex 3-fold into another.

Locally, it takes neighbourhoods that look like $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$ and applies a blowdown map π, before smoothing out the resulting singularities.

Friedman's Condition and Reid's Fantasy

In order to do this globally, we need Friedman's condition.

Theorem 1.1 (R. Friedman)

A first-order deformation of X_0 smoothing the singularities $s_i = \pi(E_i)$ exists if and only if there exist $\lambda_i \neq 0$ such that

$$\sum_i \lambda_i [E_i] = 0 \text{ in } H^2(\widehat{X}, \mathbb{R}). \quad (1)$$

Kawamata–Tian show that if we have the $i\partial\overline{\partial}$-lemma, we get genuine smoothings from the first-order ones.

If we start with a 3-fold with trivial canonical bundle, the resulting manifolds still have trivial canonical bundle.

Fantasy 1.2 (Reid)

All Calabi–Yau 3-folds are connected by a sequence of conifold transitions.

Topological Changes

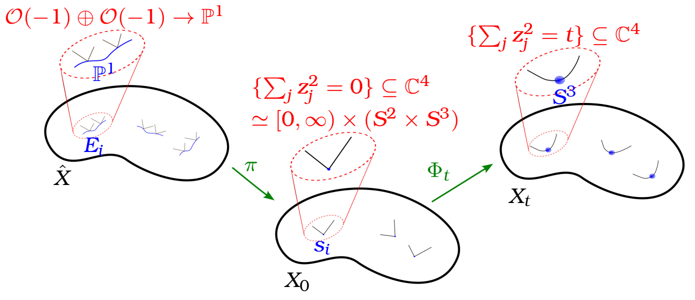

Figure: A Conifold Transtion

Topologically, a conifold transition contracts 2-cycles from the small resolution and generates 3-cycles on the smoothing.

In particular, if we contract N curves with k linearly independent curves

$$b_2(X_t) = b_2(\widehat{X}) - k, \qquad b_3(X_t) = b_3(\widehat{X}) + 2(N-k). \tag{2}$$

An Example

Let $\widehat{X}$ be a quintic 3-fold in $\mathbb{P}^4$ and so $b_2(\widehat{X}) = 1$. If we pick 2 linearly dependent curves E_1 and E_2 and apply a conifold transition, the resulting smoothing X_t has $b_2(X_t) = 0$.

The Kähler conditon is NOT preserved.

This implies that our main objects of study should include those non-Kähler manifolds obtained from Kähler ones through conifold transitions.

Generalizing the Kähler Condition

The Hull–Strominger System

It is conjectured that the general framework for compact non-Kähler geometry should involve not 1, but 2 metrics.

Let X be a Calabi–Yau 3-fold and $E \to X$ a holomorphic vector bundle. We have a nowhere vanishing holomorphic $(3,0)$-form Ω.

For a fixed constant $\alpha' \in \mathbb{R}$, we have the Hull–Strominger system which wants a pair of metrics H on E and ω on X such that

$$F_H \wedge \omega^2 = 0, \tag{3}$$

$$i\partial\overline{\partial}\omega = \frac{\alpha'}{4}\Big(\mathrm{tr}\,(\mathrm{Rm}_\omega \wedge \mathrm{Rm}_\omega) - (\mathrm{tr}\, F_H \wedge F_H)\Big), \tag{4}$$

$$d(\|\Omega\|_\omega \omega^2) = 0. \tag{5}$$

If we take $E = T^{1,0}X$, and $H = g$, we see that the Hull–Strominger system generalizes the Ricci-flat Kähler condition on X.

The Hull–Strominger System II

$$F_H \wedge \omega^2 = 0, \tag{3}$$

$$i\partial\bar{\partial}\omega = \frac{\alpha'}{4}\Big(\mathrm{tr}\,(\mathrm{Rm}_\omega \wedge \mathrm{Rm}_\omega) - (\mathrm{tr}\, F_H \wedge F_H)\Big), \tag{4}$$

$$d(\|\Omega\|_\omega \omega^2) = 0. \tag{5}$$

This system of equations arises from heterotic string theory, characterized in terms of SU(3)-structures.

Looking ahead, we will see that both (3) and (5) can be solved through conifold transitions.

It is conjectured that conifold transitions preserves the solvability of the full Hull–Strominger system.

Metric Geometry on Conifold Transitions

Local Geometry

Candelas–de la Ossa Metrics

Recall our local models are

- the small resolution: $\widehat{V} = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$,
- the cone: $V_0 = \{z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\} \subseteq \mathbb{C}^4$,
- the smoothing: $V_t = \{z_1^2 + z_2^2 + z_3^2 + z_4^2 = t\} \subseteq \mathbb{C}^4$.

Candelas–de la Ossa have constructed Ricci-flat Kähler metrics on each of these spaces which will be the basis of our model local geometry.

On the Small Resolution

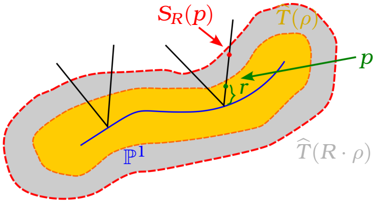

Figure: Local Small Resolution Model

On $\widehat{V} = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$, we have a "radius" function r that measures the distance from a point to the zero section with respect to the Fubini–Study metric $\widehat{\omega}_{FS}$.

In this sense, we can think of local neighbourhoods containing the $\mathbb{P}^1$ as "tubes" of a certain radius and we can scale points up and down their respective fibres using this radius function.

On the Small Resolution II

Candelas–de la Ossa considered metrics of the form

$$\widehat{\omega}_{co,a} = i\partial\overline{\partial} f_a(r) + 4a^2 \widehat{\omega}_{FS} \tag{6}$$

where a is some parameter.

If we impose the Ricci-flat Kähler condition, we end up with a DE that determines the functions f_a. This also gives a nice scaling property

$$f_a(r) = a^2 f_1\left(\frac{r}{a}\right). \tag{7}$$

From this, we see that this family of metrics depends smoothly on the parameter a.

On the Cone

After identifying via the blowdown map, the function r becomes the function $r = \|z\|^{\frac{1}{3}}$ on $V_0 = \{z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\}$.

Further, as $a \to 0$, the metrics $\widehat{\omega}_{co,a}$ approach the cone metric $\frac{1}{2}\omega_{co,0} = i\partial\overline{\partial}r^2$ on compact sets away from the singularities.

This metric turns out to be a cone metric over the link $S^2 \times S^3$ in the sense that

$$g_{co,0} = dr \otimes dr + r^2 \cdot g_{S^2 \times S^3}, \tag{8}$$

and is well-behaved away from the singularity.

On the Smoothing

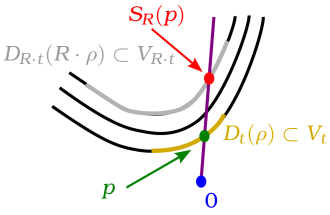

Figure: Local Smoothing Model

On $V_t = \{z_1^2 + z_2^2 + z_3^2 + z_4^2 = t\}$ we have the same "radius" function r and by scaling this radius, we can move between the spaces V_t for $t \neq 0$.

On these spaces, the Candelas–de la Ossa Ansatz is

$$\omega_{co,t} = i\partial\bar{\partial} f_t(r), \tag{9}$$

where t is some parameter. Imposing the Ricci-flat Kähler condition yields a similar scaling result to the small resolution.

On the Smoothing II

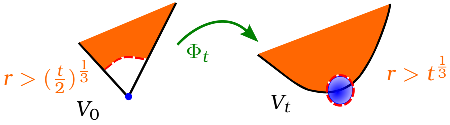

Figure: The Diffeomorphism Φ_t

To compare these metrics to the cone, we need to pull them back via some maps. Scaling the metric doesn't work so we need a more complicated map.

Define the maps $\Phi_t(z)\colon \mathbb{C}^4 \backslash \{0\} \to \mathbb{C}^4$

$$\Phi(z) = z + \frac{t\overline{z}}{2\|z\|^2}. \tag{10}$$

These map V_0 to V_t and (after cutting out certain sets) are diffeomorphisms.

After pullback, we get a similar limiting behaviour of metrics away from the singularities:

$$\Phi_t^*(g_{co,t}) \to g_{co,0}. \tag{11}$$

Properties

Here is a summary of a couple of important properties of the Candelas–de la Ossa metrics.

1. Normalization:
   - $\widehat{g}_{co,a} = a^2 \cdot S^*_{a^{-1}}(\widehat{g}_{co,1})$,
   - $g_{co,t} = t^{\frac{2}{3}} \cdot S^*_{t^{-\frac{1}{3}}}(g_{co,1})$,

2. Asymptotically Conical Decay:
   - $|(\pi^{-1})^*(\widehat{g}_{co,a}) - g_{co,0}|_{g_{co,0}} \leq Ca^2 r^{-2}$,
   - $|(\Phi_t)^*(g_{co,t}) - g_{co,0}|_{g_{co,0}} \leq Ctr^{-3}$.

Balanced Metrics

Fu–Li–Yau Metrics

Suppose we begin with a Kähler Calabi–Yau 3-fold $\widehat{X}$ and consider its blowdown $X_0 = \pi(\widehat{X})$.

In line with the study of balanced metrics ($d\omega^2 = 0$) in complex geometry initiated by Michelsohn, Fu–Li–Yau constructed balanced metrics which are close to (a multiple of) the Candelas–de la Ossa metrics near the singularities/ vanishing cycles.

This was done via a gluing construction and heavily utilizes the local model geometry to maintain positivity of the forms involved.

On the Small Resolution and Conifold

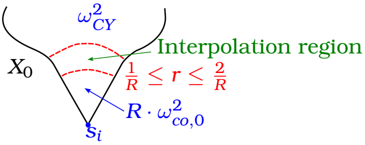

Figure: The Fu–Li–Yau Gluing Construction

The general process finds a gluing region and constructs a form Ω_0 interpolates between the squared metrics $\omega_{co,0}^2$ and ω_{CY}^2.

A similar process can be done on the original manifold $\widehat{X}$ with the Candelas–de la Ossa metrics $\widehat{\omega}_{co,a}$ and the Calabi–Yau metric $\widehat{\omega}_{CY}$.

The gluing region and cutoff function is independent of the parameter a and is also smooth in a.

On the Smoothing

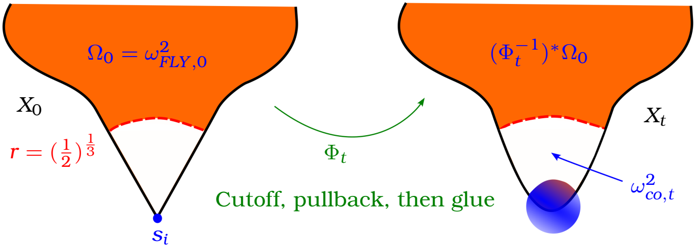

Figure: The Fu–Li–Yau Gluing Construction

In order to get balanced metrics on the smoothing, we use the maps Φ_t to push the squared Fu–Li–Yau metric Ω_0 on the singular space X_0 onto X_t away from the singularities/ vanishing cycles and glue them to (a multiple of) the squared Candelas–de la Ossa metrics $\omega^2_{co,t}$.

On the Smoothing II

So far, this only produces a form on each X_t since Φ_t is only diffeomorphic onto its image, not biholomorphic. In order to obtain something amenable to the changing complex structure, we must project to the $(2,2)$ component. This gives a positive $(2,2)$ form which we can take the square root of to get an auxiliary Hermitian metric ω_t.

We now have something of the right type, but may not be closed so a perturbation term γ_t needs to be added. This extra perturbation term is related to a solution of the Kodaira–Spencer operator and is shown to be small enough to maintain positivity of the overall form.

These processes are done smoothly in the parameter t.

Properties

Like the Candelas–de la Ossa metrics, there are two properties of the Fu–Li–Yau balanced metrics that we note in particular

1. Local Model: Around each $(-1,-1)$-curve/ vanishing sphere
   - $\widehat{g}_{FLY,a} = R \cdot \widehat{g}_{co,a}$,
   - $|g_{FLY,t} - c \cdot g_{co,t}|_{g_{co,t}} \leq Ct^{\frac{2}{3}}$,

2. Uniform Convergence: On compact sets away from the singularities
   - $(\pi^{-1}) * \widehat{g}_{FLY,a} \to g_{FLY,0}$ uniformly as $a \to 0$,
   - $(\Phi_t)^* g_{FLY,t} \to g_{FLY,0}$ uniformly as $t \to 0$.

Hermitian Yang–Mills Metrics

Collins–Picard–Yau Metrics

Recall our setup of a Kähler Calabi–Yau 3-fold $\widehat{X}$ and the conifold transition $\widehat{X} \to X_0 \rightsquigarrow X_t$. We also suppose that $\widehat{X}$ is simply connected.

Collins–Picard–Yau constructed families of Hermitian Yang–Mills metrics on the tangent bundles with respect to the Fu–Li–Yau balanced metrics.

On the Small Resolution

The simply connected condition gives a stability condition with respect to the original Ricci-flat Kähler metric $\widehat{\omega}_{CY}$:

$$\frac{1}{\operatorname{rk} F}\int_{\widehat{X}} c_1(F)\wedge \widehat{\omega}_{CY}^2 < 0, \tag{12}$$

for each torsion-free coherent subsheaf $F \subset T^{1,0}\widehat{X}$.

On the small resolution $\widehat{X}$, the Fu–Li–Yau construction is performed such that $[\widehat{\omega}_{FLY,a}^2] = [\widehat{\omega}_{CY}^2]$ and so this is passed onto the Fu–Li–Yau metrics $\widehat{\omega}_{FLY,a}$:

$$\frac{1}{\operatorname{rk} F}\int_{\widehat{X}} c_1(F)\wedge \widehat{\omega}_{FLY,a}^2 < 0, \tag{13}$$

Using an analog of the Donaldson–Uhlenback–Yau Theorem due to Li–Yau, we obtain Hermitian Yang–Mills metrics $\widehat{H}_a$ with respect to $\widehat{\omega}_{FLY,a}$.

On the Conifold and the Smoothing

After normalization, Collins–Picard–Yau show that a limiting metric Hermitian Yang–Mills metric H_0 with respect to $\omega_{FLY,0}$ can be constructed on X_0. This is done by adapting a C^0-estimate calculation by Uhlenbeck–Yau.

The process to obtain metrics on the smoothing is similar to that of Fu–Li–Yau. We can pull the metric H_0 onto X_t via the map Φ_t and glue them to the Candelas–de la Ossa metrics. These metrics then need to be perturbed to fulfill the desired properties.

Properties

The Hermitian Yang–Mills metrics on both the small resolution and the smoothing also satisfy a couple of important properties

1. Uniform Equivalence:
   - $C^{-1} \cdot \widehat{g}_{FLY,a} \leq \widehat{H}_a \leq C \cdot \widehat{g}_{FLY,a}$,
   - $C^{-1} \cdot g_{FLY,t} \leq H_t \leq C \cdot g_{FLY,t}$.
2. Uniform Convergence: On compact sets away from the singularities
   - $(\pi^{-1})^* \widehat{H}_a \to H_0$ uniformly as $a \to 0$,
   - $(\Phi_t)^* H_t \to H_0$ uniformly as $t \to 0$.

Gromov–Hausdorff Convergence

Families of Distance Spaces

We have sequences of distance spaces satisfying preserved properties through conifold transitions:

- $(\widehat{X}, \widehat{g}_{FLY,a})$ and $(X_t, g_{FLY,t})$ satisfy the balanced condition,
- $(\widehat{X}, \widehat{H}_a)$ and (X_t, H_t) satisfy the Hermitian Yang–Mills condition.

These are bridged by singular spaces $(X_0, g_{FLY,0})$ and (X_0, H_0) respectively with Riemannian metrics (except on their singular sets which are a finite set of points).

Our Riemannian metrics are all locally uniformly equivalent to the cone metric $dr \otimes dr + r^2 \cdot g_{S^2 \times S^3}$. We can extend the Riemannian metrics by 0 to the singularities.

This breaks positivity, but does not affect lengths of curves, since the singularities are isolated points and so we get bona fide distance functions on X_0. In this way, we can also view $(X_0, g_{FLY,0})$ and (X_0, H_0) as distance spaces.

Gromov–Hausdorff Topology

The space of compact distance spaces (modulo isometry) $\mathcal{M}$ can be endowed with a topology. We can understand this topology at the level of maps between these spaces.

Definition 3.1

A map $f : X \to Y$ between two compact distance spaces is an ϵ-isometry if

- $|d_X(p, q) - d_Y(f(p), f(q))| < \epsilon$ for all $p, q \in X$,
- $Y \subseteq B_\epsilon(f(X))$.

Definition 3.2

The Gromov–Hausdorff distance d_{GH} between two compact distance spaces is

$$\begin{aligned} d_{GH}((X, d_X), (Y, d_Y)) = \inf\{\epsilon > 0 \mid \text{There exist } \epsilon - \text{isometries} \\ f_1 : (X, d_X) \to (Y, d_Y), f_2 : (Y, d_Y) \to (X, d_X)\}. \end{aligned} \tag{14}$$

In general, its enough to only show an ϵ-isometry from X to Y because we can construct a 3ϵ-isometry in the other direction.

Conifold Transitions are a Continuous Process (Regular Case)

We have the following result about conifold transitions from an initially Kähler Calabi–Yau 3-fold.

Theorem 3.3 (B. Friedman–Picard–S.)

Each of the maps

- $(0,1] \to \mathcal{M} \colon a \mapsto (\widehat{X}, \widehat{g}_{FLY,a})$,
- $(0,1] \to \mathcal{M} \colon a \mapsto (\widehat{X}, \widehat{H}_a)$,
- $\Delta \backslash \{0\} \to \mathcal{M} \colon t \mapsto (X_t, g_{FLY,t})$,
- $\Delta \backslash \{0\} \to \mathcal{M} \colon t \mapsto (X_t, H_t)$,

are continuous in the Gromov–Hausdorff topology.

Conifold Transitions are a Continuous Process (Singular Case)

The more interesting case is what happens as the parameters a and t approach 0.

Theorem 3.4 (B. Friedman–Picard–S.)

We have the following limits in the Gromov–Hausdorff topology:

- $(\widehat{X}, \widehat{g}_{FLY,a}) \to (X_0, g_{FLY,0}) \leftarrow (X_t, g_{FLY,t})$ *as* $a, t \to 0$,
- $(\widehat{X}, \widehat{H}_a) \to (X_0, H_0) \leftarrow (X_t, H_t)$ *as* $a, t \to 0$,

Song (building on work by Rong–Zhang) proved a result similar to Theorem 3.4 in the case where the manifolds are projective and the metrics are Ricci-flat Kähler.

The Regular Case

The Small Resolution

Sketch of Proof ($a \mapsto (\widehat{X}_a, \widehat{g}_{FLY,a})$)

The idea here is that Gromov–Hausdorff continuity is a weaker notion than uniform convergence of Riemannian metrics.

To show continuity overall, we pick a point $b \in (0, 1]$ along the path and use the corresponding metric $\widehat{g}_{FLY,b}$ as a reference metric.

The squared metrics $\widehat{\omega}^2_{FLY,a}$ have expressions smooth in a and $p \in \widehat{X}$ and so $|\widehat{\omega}^2_{FLY,a} - \widehat{\omega}^2_{FLY,b}|$ is smooth in a and p.

The Small Resolution II

Sketch of Proof ($a \mapsto (\widehat{X}_a, \widehat{g}_{FLY,a})$ (cont'd))

Pick some compact interval $I \subseteq (0,1]$ containing b. Then $\nabla|\widehat{\omega}^2_{FLY,a} - \widehat{\omega}^2_{FLY,b}|$ is smooth on $I \times \widehat{X}$. Compactness of $I \times \widehat{X}$ gives uniform boundedness of the covariant derivative on I.

By Arzelà–Ascoli, the convergence $\widehat{\omega}^2_{FLY,a} \to \widehat{\omega}^2_{FLY,b}$ is uniform. Taking a square root of these forms is a continuous process so the metrics $\widehat{g}_{FLY,a}$ converge uniformly to $\widehat{g}_{FLY,b}$ with respect to $\widehat{g}_{FLY,b}$.

This ultimately tells us that the identity map on $\widehat{X}$ is an ϵ-isometry for a sufficiently close to b.

The Smoothing

Sketch of Proof ($t \mapsto (X_t, g_{FLY,t})$)

We have a couple of added difficulties here:

- the Fu–Li–Yau metrics on the smoothings involve a pullback and also a perturbation;
- the metrics $g_{FLY,t}$ all lie on different manifolds X_t, but we have diffeomorphisms between them.

Because of this, we need to be more careful. As before, fix some $s \in (0, 1]$ and use g_s as a reference metric. Let $F_t : X_s \to X_t$ be the family of diffeomorphisms.

We can write $\omega^2_{FLY,t} = \omega^2_t + \gamma_t$, where ω_t is the auxiliary Hermitian metric and γ_t is the perturbation term in the Fu–Li–Yau construction.

The auxiliary metrics ω_t (after pullback to X_s) are managed in a similar manner to the first path and so $F_t^* \omega_t^2 \to \omega_s^2$ uniformly with respect to ω_s.

The Smoothing II

Sketch of Proof ($t \mapsto (X_t, g_{FLY,t})$ (cont'd))

The extra terms γ_t satisfy a differential equation for the Kodaira–Spencer operator E_t. In particular, each γ_t solves

$$E_t(\gamma_t) = \overline{\partial}\omega_t^2, \tag{15}$$

where

$$E_t = \partial\overline{\partial}\overline{\partial}^\dagger\partial^\dagger + \partial^\dagger\overline{\partial}\overline{\partial}^\dagger\partial + \partial^\dagger\partial, \tag{16}$$

with respect to ω_t.

The idea here is that the RHS of (15), the complex structure on X_t and also the auxiliary metrics ω_t all vary smoothly in t and so γ_t must also do the same (up to adding elements in the kernel).

By our assumptions on the initial manifold, we can show that each E_t has trivial kernel and so each γ_t is determined uniquely.

The Smoothing III

Sketch of Proof ($t \mapsto (X_t, g_{FLY,t})$ (cont'd) (cont'd))

From what we learned about the auxiliary metrics ω_t, we can uniform Schauder estimates on X_t for t close to s:

$$\|\gamma_t\|_{C^{4,\alpha}} \leq C \cdot (\|\gamma_t\|_{C^0} + \|E_t(\gamma_t)\|_{C^{4,\alpha}}). \tag{17}$$

Using Arzelà–Ascoli, we can show that $F_t^*\gamma_t$ converge uniformly to a limit which we can show to be γ_s. The idea here is that if we assume otherwise, we can pick a convergent subsequence which violates one of our other established bounds.

Combining the two parts, we get that $F_t^*\omega_{FLY,t}^2 \to \omega_{FLY,s}^2$ uniformly and we can take a square root again to get the result.

The Hermitian Yang–Mills Metrics

The paths using the Hermitian Yang–Mills metrics can be proven using the same ideas.

They satisfy some uniform equivalence conditions and some standard derivative estimates and so using the same Arzelà–Ascoli trick and uniqueness and normalization, convergence of our sequences must be uniform.

The Singular Case

General Idea

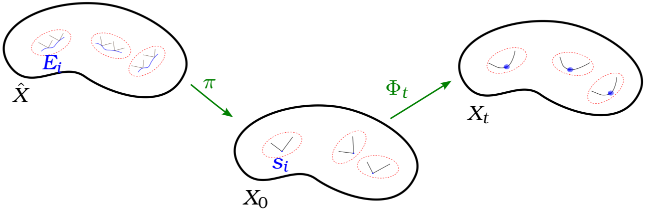

Figure: General Idea

We have natural maps between our spaces that we want to show are ϵ-isometries.

Since everything away from the contracted curves/ spheres/ singularities are well-behaved, we want to show that the local models "can be made arbitrarily small". There is a similar idea in work of Song–Weinkove on contracting exceptional divisors by the Kähler–Ricci flow.

General Idea II

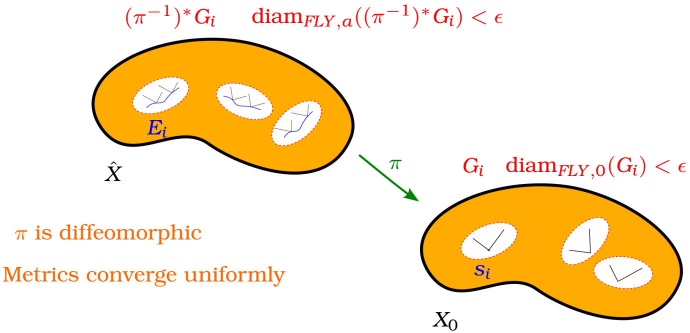

Figure: “Small” Local Geometry

Bounds on the Small Resolution

We want to get a handle of the diameter of a "tube" $\widehat{T}(\delta) = \{r \leq \delta\}$ around a contracted $\mathbb{P}^1 \subseteq \widehat{V}$ with respect to the Fu–Li–Yau metric $\widehat{g}_{FLY,a}$.

The Fu–Li–Yau metric is a multiple of the Candelas–de la Ossa metric near the contracted curves on the small resolution and so we just work with those metrics instead.

1. Normalization: $\widehat{g}_{co,a} = a^2 \cdot S^*_{a^{-1}}(\widehat{g}_{co,1})$,
2. Asymptotically Conical Decay: $|(\pi^{-1})^*(\widehat{g}_{co,a}) - g_{co,0}|_{g_{co,0}} \leq Ca^2 r^{-2}$.

We can find a constant K such that

$$|(\pi^{-1})^*(\widehat{g}_{co,a}) - g_{co,0}|_{g_{co,0}} \leq \frac{1}{2} \tag{18}$$

when $r > aK$.

Estimating the Diameter

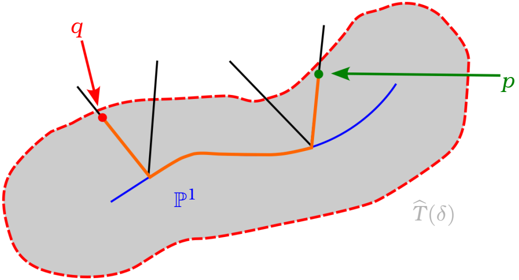

Figure: Path Connectedness

We estimate the diameter of this set by considering paths between general points. The paths we look at in particular move down along the fiber, along the zero section $\mathbb{P}^1$ and up along another fiber.

Splitting the Tube

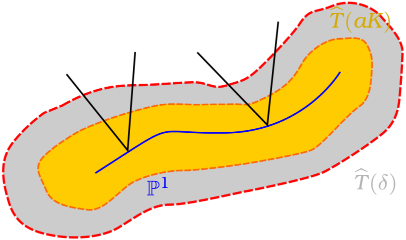

Figure: Splitting Tubes

We ultimately want uniform bounds on the diameter of this set.

For $a \leq \frac{\delta}{K}$, we break our "tube" $\widehat{T}(\delta) = \{r \leq \delta\}$ into two parts:

- a smaller "tube" $\widehat{T}(aK)$ which we will pull back and compare to $\widehat{g}_{co,1}$,
- and an "annulus" $\widehat{T}(\delta) \backslash \widehat{T}(aK)$ which we will compare to $g_{co,0}$.

Tubular Bounds

Using the normalization property, we can compare lengths of curves using the metric $\widehat{g}_{co,a}$ and using the metric $\widehat{g}_{co,1}$.

Given a curve $\gamma\colon [0,1] \to \widehat{T}(aK)$ we have

$$\begin{aligned}
\widehat{L}_{co,a}(\gamma) &= \int_0^1 \sqrt{\widehat{g}_{co,a}\Big(\dot\gamma(s),\dot\gamma(s)\Big)}\,ds \\
&= \int_0^1 \sqrt{a^2\cdot S^*_{a^{-1}}(\widehat{g}_1)\Big(\dot\gamma(s),\dot\gamma(s)\Big)}\,ds \\
&= a\cdot\int_0^1 \sqrt{\widehat{g}_{co,1}\Big((S_{a^{-1}})_*\dot\gamma(s),(S_{a^{-1}})_*\dot\gamma(s)\Big)}\,ds \\
&= a\cdot\widehat{L}_{co,1}(S_{a^{-1}}\circ\gamma),
\end{aligned}$$

This means that $\widehat{\operatorname{diam}}_{co,a}(\widehat{T}(aK)) = a\cdot\widehat{\operatorname{diam}}_{co,1}(\widehat{T}(K))$.

The set $\widehat{T}(K)$ is independent of a and is compact and so $\widehat{\operatorname{diam}}_{co,1}(\widehat{T}(K))$ is a constant.

Annular Bounds

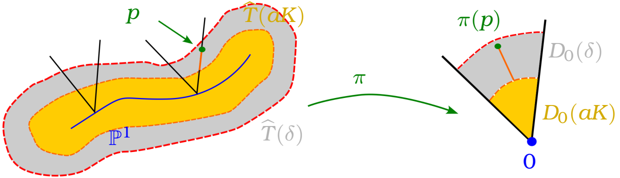

Figure: Pushing the Curve Forward

We look at the distance from a point in this set to the "tube" $\widehat{T}(aK)$. As previously mentioned, the simplest curve to consider is one along the fibre towards the $\mathbb{P}^1$.

Pick a point p with $aK < r(p) = \rho \leq \delta$. We can describe our desired curve in coordinates and use the blowdown map to push this to the cone V_0.

Annular Bounds II

We can use the cone metric to measure this new curve

$$L_{co,0}(\gamma) = (\rho - aK). \tag{19}$$

We can also use the asymptotically conical decay estimate to get that

$$|\widehat{L}_{co,a}(\widehat{\gamma}) - L_{co,0}(\gamma)| \leq Ca \cdot \left(\frac{1}{K} - \frac{a}{\rho}\right). \tag{20}$$

Combining these, we get that

$$\widehat{L}_{co,a}(\widehat{\gamma}) \leq Ca \cdot \left(\frac{1}{K} - \frac{a}{\rho}\right) + (\rho - aK). \tag{21}$$

and for $0 < a \leq \frac{\delta}{K}$ and $aK < \rho \leq \delta$ the RHS is bounded by $C \cdot (\delta - aK)$.

This gives an upper bound on the distance from a point in the “annulus” $\widehat{T}(\delta) \backslash \widehat{T}(aK)$ to the “tube” $\widehat{T}(aK)$.

Combined Bound

If we combine our two bounds, we get

$$\widehat{\text{diam}}_{co,a}(\widehat{T}(\delta)) \leq a \cdot \widehat{\text{diam}}_{co,1}(\widehat{T}(K)) + 2C \cdot (\delta - aK), \tag{22}$$

which for fixed δ and K, is uniformly bounded for $0 < a \leq \frac{\delta}{K}$.

Rewriting this gives

$$\widehat{\text{diam}}_{co,a}(\widehat{T}(\delta)) \leq C \cdot \delta \tag{23}$$

for all $0 < a \leq \frac{\delta}{K}$ where C is a uniform constant.

As such, we have shown that for any $\epsilon > 0$, we can find some $\delta = \frac{\epsilon}{C} > 0$ and some $a_0 \in (0, 1]$ such that

$$\widehat{\text{diam}}_{co,a}(\widehat{T}(\delta)) \leq \epsilon \tag{24}$$

for all $a \leq a_0 \leq \frac{\delta}{K} = \frac{\epsilon}{CK}$.

Bounds on the Cone

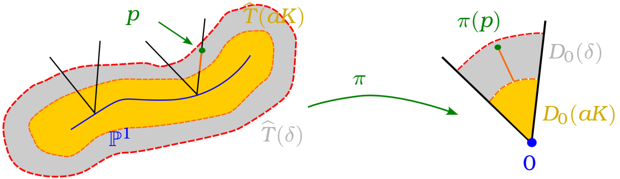

Figure: Pushing the Curve Forward

We can check that the blowdown map π maps the "tube" $\widehat{T}(\delta)$ to the "disc" $D_0(\delta) = \{r \leq \delta\}$ around $0 \in V_0$.

Using the cone metric, we see that

$$\mathrm{diam}_{co,0}(D_0(\delta)) \leq 2\delta. \tag{25}$$

So, for any $\epsilon > 0$ we can find some $\delta = \frac{\epsilon}{2} > 0$ such that

$$\mathrm{diam}_{co,0}(D_0(\delta)) \leq \epsilon. \tag{26}$$

ϵ-Isometries

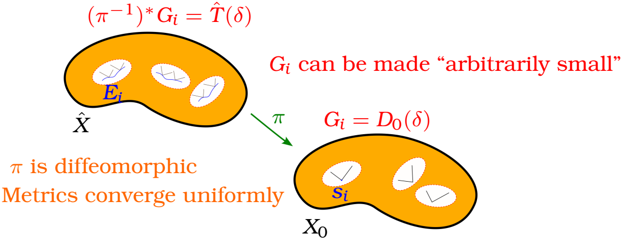

Figure: An ϵ-Isometry

In essence, we have shown that for any $\epsilon > 0$ we have a set $G_i = D_0(\delta)$ containing each singularity s_i with $\text{diam}_{FLY,0}(G_i) < \epsilon$ with preimage $\pi^{-1}(G_i) = \widehat{T}(\delta)$ containing the $(-1,-1)$-curve E_i and $\widehat{\text{diam}}_{FLY,a}(\pi^{-1}(G_i)) < \epsilon$ for $a \leq \frac{\delta}{K}$.

Combined with uniform convergence away from these sets, we see that π is an ϵ-isometry for sufficiently small a.

The Other Cases

A similar idea works for the smoothings.

We can get sufficiently small sets with respect to the Candelas–de la Ossa metrics $\omega_{co,t}$, but these are not the Fu–Li–Yau metrics yet.

The difference between the Fu–Li–Yau metrics and the Candelas–de la Ossa metrics is small, so the above can be achieved with respect to the Fu–Li–Yau metrics as well.

To get the Hermitian Yang–Mills metrics, we use uniform equivalence of the metrics to the Fu–Li–Yau metrics to get the analogous result.

These ϵ-isometries imply the Gromov–Hausdorff convergence of our spaces.

Thank you for your attention.