Banff, Spinorial and Octonionic Aspects of G_2 and Spin(7) Geometry

Flows of G_2 Structures Associated to Calabi–Yau Manifolds

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Overview

Goal

Establish a correspondence between the Laplacian flow and coflow on torus bundles over Calabi–Yau 2- and 3-folds with Monge–Ampère flows on the base.

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This is joint work with Sébastien Picard.

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A Calabi–Yau manifold has the following properties:

- the canonical bundle K_X is trivial,
- the first Chern class $c_1(X)$ vanishes,
- the Ricci-form $\mathrm{Ric}(\omega, J)$ is given by $2i\partial\overline{\partial}(\log |\Omega|_\omega)$ and it vanishes if and only if $|\Omega|_\omega$ is constant.

Yau's Theorem

Theorem (Yau)

Let (X, ω) be a compact Kähler manifold with $c_1(X) = 0$ and let $F \colon X \to \mathbb{R}$ be a function such that

$$\int_X e^F \omega^n = \int_X \omega^n.$$

Then there is a smooth function $u \colon X \to \mathbb{R}$, unique up to the addition of a constant, such that

$$\omega + i\partial\bar{\partial}u > 0 \text{ and } (\omega + i\partial\bar{\partial}u)^n = e^F \omega^n.$$

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Yau's theorem and its proof involved the solving of complex Monge–Ampère equations, which have since been studied extensively.

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Theorem (Picard–Zhang)

Let (X, ω) be a compact Kähler manifold. Let $a\colon X \to \mathbb{R}$ be a function and let $H\colon \mathbb{R}^+ \to \mathbb{R}$ be a smooth function with $H' > 0$. Then there exists a solution u_t to the parabolic complex Monge–Ampère equation

$$\frac{\partial}{\partial t} u_t = H\Big(e^{-a}\frac{\det(\omega + i\partial\overline{\partial}u_t)}{\det\omega}\Big), \qquad \omega + i\partial\overline{\partial}u_t > 0, \qquad u_0 = 0.$$

This solution exists for all time t. Moreover, the metrics $\widetilde{\omega}_t = \omega + i\partial\overline{\partial}u_t$ converge in each $C^k(X, g)$-norm to a limiting metric $\omega' \in [\omega]$.

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When X is a Calabi–Yau manifold, the limiting metric is the Calabi–Yau metric ω_{CY}.

Monge–Ampère Flows

Certain choices of the functions a and H give familiar special cases:

- Kähler–Ricci flow ($H(\rho) = \log \rho$),
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We have two particular cases of importance:

- MA$^{\frac{1}{3}}$ flow ($a = 2\log|\Omega|_\omega$, $H = 6K\rho^{\frac{1}{3}}$):

$$\frac{\partial}{\partial t}u_t = 6K\Big(e^{-2\log|\Omega|_\omega}\frac{\det(\omega + i\partial\bar{\partial}u_t)}{\det\omega}\Big)^{\frac{1}{3}},$$

- Kähler–Ricci flow ($a = 2\log|\Omega|_\omega$, $H = 2K\log\rho$):

$$\frac{\partial}{\partial t}u_t = 2K\log\Big(\frac{\det(\omega + i\partial\bar{\partial}u_t)}{\det\omega}\Big) - 2K\log|\Omega|^2_\omega.$$

Uniform Estimates

The evolving metrics $\widetilde{g}_t$ from a Monge–Ampère flow satisfy uniform estimates: There exist positive constants C and C_k such that

$$C^{-1} \cdot g \leq \widetilde{g}_t \leq C \cdot g \text{ and } |\nabla_g^k \widetilde{\omega}_t|_g \leq C_k.$$

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We also have exponential convergence of the flow: There exist positive constants C_k and λ_k such that

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These estimates were previously known for certain special cases like the Kähler–Ricci flow (Cao, Phong–Sturm).

G_2 Structures

Definition

A 3-form φ on a 7-fold M is called a G_2 structure if for each $p \in M$ and non-zero $Y_p \in T_pM$,

$$(Y_p \lrcorner \varphi) \wedge (Y_p \lrcorner \varphi) \wedge \varphi \neq 0.$$

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A G_2 structure φ induces a metric g_7 and a Riemannian volume form vol_7 by the relation

$$-\frac{1}{6}(Y \lrcorner \varphi) \wedge (Z \lrcorner \varphi) \wedge \varphi = g_7(Y, Z) \cdot \mathrm{vol}_7.$$

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We in turn obtain an induced Hodge star operator $\star_7$ and a dual 4-form $\psi = \star_7 \varphi$.

Torsion

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$$d\varphi = \tau_0\psi + 3\tau_1 \wedge \varphi + \star\tau_3,$$

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Types of G_2 Structure

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When φ is torsion-free, the Riemannian holonomy of the metric g is contained in the group G_2 and the metric g is Ricci-flat (Fernández–Gray).

Laplacian Flow

Definition

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Theorem (Bryant–Xu)

Let φ be a closed G_2 structure on a compact 7-fold M. Then, the Laplacian flow with initial condition φ has a unique solution for a short-time $[0, T]$ with T depending on φ.

Laplacian Coflow

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Unlike the Laplacian flow, short-time existence and uniqueness for the Laplacian coflow is not known.

G_2 Structures from Calabi–Yau 2-Folds

Let (X^4, ω, Ω) be a Calabi–Yau 2-fold and choose:

- a nowhere-vanishing complex function F on X^4,
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We can define a G_2 structure φ on $M^7 = T^3 \times X^4$ by setting

$$\varphi = -G dr^1 \wedge dr^2 \wedge dr^3 + dr^1 \wedge G\omega + dr^2 \wedge Re\left(\frac{F}{|\Omega|_\omega}\Omega\right) + dr^3 \wedge Im\left(\frac{F}{|\Omega|_\omega}\Omega\right).$$

Here r^1, r^2, and r^3 denote the angle coordinates on T^3.

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The associated metric and volume form are

$$\begin{aligned}g_7 = {} & 2^{\frac{4}{3}}|F|^{-\frac{4}{3}}G^2(dr^1)^2 + 2^{-\frac{2}{3}}|F|^{\frac{2}{3}}(dr^2)^2 \\ & + 2^{-\frac{2}{3}}|F|^{\frac{2}{3}}(dr^3)^2 + 2^{-\frac{2}{3}}|F|^{\frac{2}{3}}g_4,\end{aligned}$$

and

$$\mathrm{vol}_7 = 2^{-\frac{4}{3}}|F|^{\frac{4}{3}}G dr^1 \wedge dr^2 \wedge dr^3 \wedge \mathrm{vol}_4.$$

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If $\alpha \in \Omega^k(X^4)$ is a k-form, then the Hodge star acts as

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\star_7 \alpha &= (-1)^k 2^{(-\frac{4}{3}+\frac{2}{3}k)} |F|^{(\frac{4}{3}-\frac{2}{3}k)} G dr^1 \wedge dr^2 \wedge dr^3 \wedge \star_4 \alpha, \\
\star_7 (dr^1 \wedge \alpha) &= 2^{(-\frac{8}{3}+\frac{2}{3}k)} |F|^{(\frac{8}{3}-\frac{2}{3}k)} G^{-1} dr^2 \wedge dr^3 \wedge \star_4 \alpha, \\
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Lastly, we can check that the dual 4-form ψ is

$$\begin{aligned}
\psi = &-2^{-\frac{4}{3}} |F|^{\frac{4}{3}} \cdot \frac{1}{2} \omega^2 + 2^{-\frac{4}{3}} |F|^{\frac{4}{3}} dr^2 \wedge dr^3 \wedge \omega \\
&+ 2^{\frac{2}{3}} |F|^{-\frac{2}{3}} G dr^3 \wedge dr^1 \wedge Re\left(\frac{F}{|\Omega|_\omega} \Omega\right) + 2^{\frac{2}{3}} |F|^{-\frac{2}{3}} G dr^1 \wedge dr^2 \wedge Im\left(\frac{F}{|\Omega|_\omega} \Omega\right).
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The associated metric and volume form are

$$g_7 = 4|F|^{-\frac{4}{3}} G^2 (dr)^2 + \frac{1}{2}|F|^{\frac{2}{3}} g_6,$$

and

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Closed G_2 Structures

With an appropriate choice of the functions F and G, we can obtain closed G_2 structures on the product manifolds. In particular $F = |\Omega|_\omega$ and $G = 1$ yields the forms

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$$\varphi = -dr^1 \wedge dr^2 \wedge dr^3 + dr^1 \wedge \omega + dr^2 \wedge Re\left(\Omega\right) + dr^3 \wedge Im\left(\Omega\right) \text{ on } T^3 \times X^4,$$

and

$$\varphi = Re(\Omega) - dr \wedge \omega \text{ on } S^1 \times X^6.$$

Computing the Hodge Laplacians, we get

$$\Delta_d \varphi = 2^{\frac{2}{3}} \cdot \mathcal{L}_{\nabla_{(g_4)}\left(|\Omega|_\omega^{-\frac{2}{3}}\right)} \left(2dr^1 \wedge \omega - dr^2 \wedge Re\left(\Omega\right) - dr^3 \wedge Im\left(\Omega\right)\right) \text{ on } T^3 \times X^4,$$

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The Torsion Forms

In both cases, we can compute the respect torsion forms of the G_2 structures. In particular, we have that

$$\tau_0 = 0, \qquad \tau_1 = 0, \qquad \tau_3 = 0,$$

$$\tau_2 = 2^{\frac{2}{3}} \cdot \left(\nabla_{(g_4)}(|\Omega|_\omega^{-\frac{2}{3}})\right) \lrcorner \left[-2dr^1 \wedge \omega + dr^2 \wedge Re\,(\Omega) + dr^3 \wedge Im\,(\Omega)\right] \text{ on } T^3 \times X^4,$$

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The torsion forms vanish if and only if $|\Omega|_\omega$ is constant or equivalently when ω is Calabi–Yau.

Evolution Equations

If we assume that the Laplacian flow preserves the ansatz, then we have the evolution equation

$$\frac{\partial}{\partial t}\Big(- dr^1 \wedge dr^2 \wedge dr^3 + dr^1 \wedge \omega_t + dr^2 \wedge Re\left(\Omega_t\right) + dr^3 \wedge Im\left(\Omega_t\right)\Big)$$
$$= 2^{\frac{2}{3}} \cdot \mathcal{L}_{\nabla_{(g_4)_t}\left(|\Omega_t|_{\omega_t}^{-\frac{2}{3}}\right)}\Big(2dr^1 \wedge \omega_t - dr^2 \wedge Re\left(\Omega_t\right) - dr^3 \wedge Im\left(\Omega_t\right)\Big) \text{ on } T^3 \times X^4,$$

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The angle coordinates are not affected by the Lie derivatives or time derivatives. The terms involving ω_t and Ω_t are similar in both cases so we can tackle them simultaneously.

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Let h_t denote the metric in either case.

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Calabi–Yau structures satisfying the above equations will induce G_2 structures that satisfy the Laplacian flow.

Remark

A priori, it is not clear that structures satisfying the above evolution equations remain compatible as Calabi–Yau structures.

Evolution Equations

We can expand the Lie derivative terms in the evolution equations. Working on the first equation, we get

$$\frac{\partial}{\partial t}\omega_t = 2K \cdot \mathcal{L}_{\nabla_{h_t}\left(|\Omega_t|_{\omega_t}^{-\frac{2}{3}}\right)}\omega_t = 4K \cdot i\partial_t\overline{\partial}_t(|\Omega_t|_{\omega_t}^{-\frac{2}{3}}).$$

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The Lie derivative term in the second equation

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Idea

In order to address the compatibility conditions (and the fact that J_t needs to change), we can look for solutions by acting on compatible Calabi–Yau structures via a moving family of diffeomorphisms. This idea is similar to that of Fei–Phong–Picard–Zhang.

A Solution from the MA$^{\frac{1}{3}}$ Flow

Fix an initial Calabi–Yau structure (ω, Ω) on a compact Calabi–Yau n-fold X.

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The MA$^{\frac{1}{3}}$ flow then gives the existence of a solution u_t to the equation

$$\frac{\partial}{\partial t} u_t = 6K \cdot \left(e^{-2 \log |\Omega|_\omega} \frac{\det(\omega + i\partial\overline{\partial} u_t)}{\det \omega} \right)^{\frac{1}{3}}, \qquad \omega + i\partial\overline{\partial} u_t > 0, \qquad u_0 = 0.$$

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In turn, we get a family of Kähler metrics $\widetilde{\omega}_t = \omega + i\partial\overline{\partial}u_t$ which converge to the Calabi–Yau metric ω_{CY} that also satisfy

$$\frac{\partial}{\partial t}\widetilde{\omega}_t = 6K \cdot i\partial\overline{\partial}\frac{\partial}{\partial t}u_t = 6K \cdot i\partial\overline{\partial}(|\Omega|_{\widetilde{\omega}_t})^{-\frac{2}{3}}.$$

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Using the time-dependent Kähler metrics, we can define a vector field Y by

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Using Θ_t, we can pull our tensors back. The pullback structures

$$\omega_t = \Theta_t^* \widetilde{\omega}_t, \qquad \Omega_t = \Theta_t^* \Omega, \qquad J_t = \Theta_t^* J, \qquad h_t = \Theta_t^* \widetilde{h}_t,$$

remain compatible with one another as Calabi–Yau structures.

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Using DeTurck's trick, we can show that ω_t and Ω_t satisfy our desired evolution equations.

$$\frac{\partial}{\partial t}\omega_t = \frac{\partial}{\partial t}(\Theta_t^*\widetilde{\omega}_t) = \Theta_t^*(\mathcal{L}_{Y_t}\widetilde{\omega}_t) + \Theta_t^*\left(\frac{\partial}{\partial t}\widetilde{\omega}_t\right)$$

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Similarly, we can check that

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Summary

The structures (ω_t, Ω_t) satisfy the desired evolution equations

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satisfy the Laplacian flow equation.

Uniqueness of the Laplacian flow tells us that this solution is unique given the initial condition.

Coclosed G_2 Structures

Reversing our choices for F and G, we can obtain coclosed G_2 structures on the product manifolds.

$$\psi = -2^{-\frac{4}{3}} \cdot \frac{1}{2}\omega^2 + 2^{-\frac{4}{3}} dr^2 \wedge dr^3 \wedge \omega + 2^{\frac{2}{3}} dr^3 \wedge dr^1 \wedge Re\,(\Omega) + 2^{\frac{2}{3}} dr^1 \wedge dr^2 \wedge Im\,(\Omega) \text{ on } T^3 \times X^4,$$

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Computing the Hodge Laplacians, we get

$$\Delta_d \psi = 2^{\frac{2}{3}} \cdot \mathcal{L}_{\nabla_{(g_4)}(\log|\Omega|_\omega)} \Big(2^{-\frac{4}{3}} \cdot \frac{1}{2}\omega^2 - 2^{-\frac{4}{3}} dr^2 \wedge dr^3 \wedge \omega + 2^{\frac{2}{3}} dr^3 \wedge dr^1 \wedge Re\,(\Omega) + 2^{\frac{2}{3}} dr^1 \wedge dr^2 \wedge Im\,(\Omega)\Big) \text{ on } T^3 \times X^4,$$

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$$\Delta_d \psi = 2 \cdot \mathcal{L}_{\nabla_{(g_6)}(\log|\Omega|_\omega)} \Big(-2dr \wedge Im\,(\Omega) + \frac{1}{4} \cdot \frac{1}{2}\omega^2\Big) \text{ on } S^1 \times X^6.$$

The Torsion Forms

In both cases, we can compute the respect torsion forms of the G_2 structures. In particular, we have that

$$\tau_0 = 0, \qquad \tau_1 = 0, \qquad \tau_2 = 0,$$
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The torsion forms vanish if and only if $|\Omega|_\omega$ is constant or equivalently when ω is Calabi–Yau.

Evolution Equations

If we assume that the Laplacian coflow preserves the ansatz, then we have the evolution equation

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big(-2^{-\frac{4}{3}}\cdot\frac{1}{2}\omega_t^2+2^{-\frac{4}{3}}dr^2\wedge dr^3\wedge\omega_t \\
&\quad +2^{\frac{2}{3}}dr^3\wedge dr^1\wedge Re\left(\Omega_t\right)+2^{\frac{2}{3}}dr^1\wedge dr^2\wedge Im\left(\Omega_t\right)\Big) \\
&=2^{\frac{2}{3}}\cdot\mathcal{L}_{\nabla_{(g_4)_t}(\log|\Omega_t|_{\omega_t})}\Big(2^{-\frac{4}{3}}\cdot\frac{1}{2}\omega_t^2-2^{-\frac{4}{3}}dr^2\wedge dr^3\wedge\omega_t \\
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Matching terms again, we get:

$$\frac{\partial}{\partial t}\omega_t = -K \cdot \mathcal{L}_{\nabla_{h_t}(\log|\Omega_t|_{\omega_t})}\omega_t,$$

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Remark

As before, it is not clear that structures satisfying the above evolution equations remain compatible as Calabi–Yau structures.

Evolution Equations

Working with the Lie derivative terms in the evolution equations, we get

$$\frac{\partial}{\partial t}\omega_t = -K \cdot \mathcal{L}_{\nabla_{h_t}(\log|\Omega_t|_{\omega_t})}\omega_t = -2K \cdot i\partial_t\overline{\partial}_t(\log|\Omega_t|_{\omega_t}) = -K \cdot \mathrm{Ric}(\omega_t, J_t).$$

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Idea

We can again look for solutions by acting on compatible Calabi–Yau structures via a moving family of diffeomorphisms.

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Fix an initial Calabi–Yau structure (ω, Ω) on a compact Calabi–Yau n-fold X.

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The (rescaled) Kähler–Ricci flow then gives the existence of a family of Kähler metrics $\widetilde{\omega}_t$ which converge to the Calabi–Yau metric ω_{CY} that also satisfy

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We in turn obtain a 1-parameter family of diffeomorphisms such that

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We then define the pullback structures

$$\omega_t = \Theta_t^*\widetilde{\omega}_t, \qquad \Omega_t = \Theta_t^*\Omega, \qquad J_t = \Theta_t^*J, \qquad h_t = \Theta_t^*\widetilde{h}_t.$$

A Solution from the Kähler–Ricci Flow

Using DeTurck's trick, we can show that ω_t and Ω_t satisfy our desired evolution equations.

$$\frac{\partial}{\partial t}\omega_t = \frac{\partial}{\partial t}(\Theta_t^*\widetilde{\omega}_t) = \Theta_t^*(\mathcal{L}_{Y_t}\widetilde{\omega}_t) + \Theta_t^*\left(\frac{\partial}{\partial t}\widetilde{\omega}_t\right)$$

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Similarly, we can check that

$$\begin{aligned}\frac{\partial}{\partial t}\Omega_t &= \frac{\partial}{\partial t}(\Theta_t^*\Omega) = \Theta_t^*(\mathcal{L}_{Y_t}\Omega)\\ &= \mathcal{L}_{(\Theta_t^{-1})_*Y_t}(\Theta_t^*\Omega) = K\cdot\mathcal{L}_{\nabla_{h_t}(\log|\Omega_t|_{\omega_t})}\Omega_t,\end{aligned}$$

Summary

The structures (ω_t, Ω_t) satisfy the desired evolution equations

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It follows that the associated G_2 structures

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satisfy the Laplacian coflow equation.

The Story So Far

We have found a family of solutions to the Laplacian flow and coflow in terms of Calabi–Yau structures on the base manifold.

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With these ingredients, we will prove convergence of the structures (ω_t, Ω_t) and their associated G_2 structures (borrowing ideas from Lotay–Wei).

The Limit Diffeomorphism

Recall that Y_t was defined either by

$$Y_t = -K \cdot \nabla_{\widetilde{h}_t}(|\Omega|_{\widetilde{\omega}_t}|^{-\frac{2}{3}}) \text{ or } Y_t = K \cdot \nabla_{\widetilde{h}_t}(\log|\Omega|_{\widetilde{\omega}_t}|),$$

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It follows that the vector field Y_t converges to 0 exponentially fast in each $C^k(X, h)$ norm and so there exist positive constant C_k, λ_k such that

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We then see that

$$d_h(\Theta_{t_1}(p), \Theta_{t_2}(p)) \leq \int_{t_1}^{t_2} \left|\frac{\partial}{\partial t}\Theta_t(p)\right|_h dt \leq \int_{t_1}^{t_2} |Y_t|_h dt \leq C_0 \int_{t_1}^{t_2} e^{-\lambda_0 t} dt,$$

and so the maps Θ_t converge uniformly with respect to h.

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$$d_h(\Theta_{t_1}(p), \Theta_{t_2}(p)) \leq \int_{t_1}^{t_2} \left|\frac{\partial}{\partial t}\Theta_t(p)\right|_h dt \leq \int_{t_1}^{t_2} |Y_t|_h dt \leq C_0 \int_{t_1}^{t_2} e^{-\lambda_0 t} dt,$$

and so the maps Θ_t converge uniformly with respect to h.

The other uniform estimates show that the Θ_t converge in each $C^k(X, h)$-norm and so we have some limit map Θ_∞.

The Limit Diffeomorphism

Next, for each t, we have

$$\left|\frac{\partial}{\partial t}\log\left(\frac{\Omega_t\wedge\overline{\Omega}_t}{\Omega\wedge\overline{\Omega}}\right)\right| = \left|\frac{\partial}{\partial t}\left(\log\frac{\Theta_t^*(\Omega\wedge\overline{\Omega})}{\Omega\wedge\overline{\Omega}}\right)\right| = \left|\frac{1}{\Theta_t^*(\Omega\wedge\overline{\Omega})}\frac{\partial}{\partial t}\left(\Theta_t^*(\Omega\wedge\overline{\Omega})\right)\right|$$

$$= \left|\Theta_t^*\left(\frac{\mathcal{L}_{Y_t}(\Omega\wedge\overline{\Omega})}{\Omega\wedge\overline{\Omega}}\right)\right| \leq \sup_X\left|\left(\frac{\mathcal{L}_{Y_t}(|\Omega|^2_\omega \mathrm{vol})}{|\Omega|^2_\omega \mathrm{vol}}\right)\right|$$

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$$C^{-1}\cdot(\Omega\wedge\overline{\Omega}) \le \Theta_t^*(\Omega\wedge\overline{\Omega}) \le C\cdot(\Omega\wedge\overline{\Omega}),$$

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We get that $\det(\Theta_t^*)$ is uniformly bounded and this estimate can be passed to the limit Θ_∞.

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Theorem (Picard–S.)

Start the Laplacian flow with initial data

$$\varphi = -dr \wedge dr^2 \wedge dr^3 + dr^1 \wedge \omega + dr^2 \wedge Re\,(\Omega) + dr^3 \wedge Im\,(\Omega) \text{ on } T^3 \times X^4,$$

or

$$\varphi = Re\,(\Omega) - dr \wedge \omega \text{ or } S^1 \times X^6.$$

Then the Laplacian flow exists for all time t and is given by the $MA^{\frac{1}{3}}$ flow (up to diffeomorphism) and converges to a stationary point

$$\begin{aligned}\varphi_\infty = {} & -dr \wedge dr^2 \wedge dr^3 + dr^1 \wedge \Theta_\infty^* \omega_{CY} \\ & + dr^2 \wedge Re\,(\Theta_\infty^* \Omega) + dr^3 \wedge Im\,(\Theta_\infty^* \Omega) \text{ on } T^3 \times X^4\end{aligned}$$

or

$$\varphi_\infty = Re\,(\Theta_\infty^* \Omega) - dr \wedge \Theta_\infty^* \omega_{CY} \text{ on } S^1 \times X^6,$$

where Θ_∞ is a diffeomorphism on the base and ω_{CY} is the unique Ricci-flat Kähler metric in the class $[\omega]$.

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$$\psi = -2^{-\frac{4}{3}} \cdot \frac{1}{2}\omega^2 + 2^{-\frac{4}{3}} \cdot dr^2 \wedge dr^3 \wedge \omega + 2^{\frac{2}{3}} \cdot dr^3 \wedge dr^1 \wedge Re\,(\Omega) + 2^{\frac{2}{3}} \cdot dr^1 \wedge dr^2 \wedge Im\,(\Omega) \text{ on } T^3 \times X^4$$

or

$$\psi = -2 \cdot dr \wedge Im\,(\Omega) - \frac{1}{4} \cdot \frac{1}{2}\omega^2 \text{ on } S^1 \times X^6.$$

Then the Laplacian coflow exists for all time t and is given by the Kähler–Ricci flow (up to diffeomorphism) and converges to a stationary point

$$\psi_\infty = -2^{-\frac{4}{3}} \cdot \frac{1}{2}\Theta_\infty^*\omega_{CY}^2 + 2^{-\frac{4}{3}} \cdot dr^2 \wedge dr^3 \wedge \Theta_\infty^*\omega_{CY} + 2^{\frac{2}{3}} \cdot dr^3 \wedge dr^1 \wedge Re\,(\Theta_\infty^*\Omega) + 2^{\frac{2}{3}} \cdot dr^1 \wedge dr^2 \wedge Im\,(\Theta_\infty^*\Omega) \text{ on } T^3 \times X^4$$

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$$\psi_\infty = -2 \cdot dr \wedge Im\,(\Theta_\infty^*\Omega) - \frac{1}{4} \cdot \frac{1}{2}\Theta_\infty^*\omega_{CY}^2 \text{ on } S^1 \times X^6,$$

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Thank you.